Analytic Thermal Bootstrap

Iheon Kim*
Department of Physics, Pohang University of Science and Technology
(Dated: January 9, 2026)

This thesis reviews the analytic bootstrap program for conformal field theories at finite temperature, as proposed in [1]. We discuss how the analytic structure of thermal correlation functions constrains the thermal data. First, we analyze the thermal two-point function at zero spatial separation and derive an exact formula using dispersion relations. Next, we extend this analysis to non-zero spatial separation. To reconstruct the full correlator, we employ a construction that explicitly sums the result of the dispersion relation over all periodic images to enforce periodicity. We note that this approach does not automatically guarantee the clustering condition at large distances, so it must be verified on a case-by-case basis. Applying this framework, we analyze the $O(N)$ model and provide a rigorous mathematical proof for the asymptotic behavior of heavy operator coefficients. Finally, we numerically determine the thermal one-point function coefficients for the 3d Ising, XY, and Heisenberg models, reproducing established results from [2].

CONTENTS

I. INTRODUCTION

The unification of quantum mechanics and special relativity into the framework of quantum field theory (QFT) is one of the most significant achievements in modern physics. QFT provides a universal language that describes physical phenomena across vast scales, from the fundamental interactions of elementary particles to the collective behavior of condensed matter systems.

Among the various types of QFTs, Conformal Field Theories (CFTs) occupy a particularly important position. Characterized by their invariance under scale and conformal transformations, CFTs appear in various contexts of theoretical physics. They correspond to fixed points in the renormalization group (RG) flow and describe the universality classes of second-order phase transitions in statistical physics. Furthermore, through the holographic principle, a conjectured duality between a quantum gravity theory in $(d + 1)$-dimensional Anti-de Sitter (AdS) space and a d-dimensional CFT, CFTs also play a crucial role in the understanding of quantum gravity [3].

Although CFTs are defined by exact conformal symmetry, their applicability extends naturally to systems at finite temperature ($T > 0$). Understanding thermal CFTs is crucial to bridge theoretical frameworks with physical phenomena in a wide range of physical settings, among which the following two areas are of particular interest.

First, in the context of quantum criticality, experimental probes of quantum phase transitions necessarily occur

* 2heon2@postech.ac.kr

at non-zero temperatures. At quantum critical points, the characteristic energy scale vanishes, making the temperature the dominant scale. In this quantum critical region, the physics is described by a thermal CFT, where thermal fluctuations interact intricately with quantum fluctuations. Accurate theoretical predictions for this regime are essential for interpreting experimental data in condensed matter systems.

Second, in the context of the holographic principle, introducing a finite temperature to the boundary CFT corresponds to placing a black hole in the bulk AdS spacetime. Due to Hawking radiation, black holes are thermal objects, so the CFT at the boundary should inherit this thermal behavior. Thus, the study of thermal effects in strongly coupled CFTs provides a method to investigate the thermodynamics of black holes and quantum gravity [4]. In recent years, it has been demonstrated that holographic thermal two-point functions can be computed using CFT techniques, particularly the conformal bootstrap [5–7].

Initially, it was proposed in [8] that the Kubo-Martin-Schwinger (KMS) condition imposes constraints on finite-temperature dynamics. This idea was realized in [9], which computed thermal one-point function data for generalized free field theory and the $O(N)$ model at large N, and further developed a perturbation theory for thermal data in the large spin, low-twist spectrum. Subsequently, [10] computed the thermal one-point function data for the 3d Ising model. More recently, [2] numerically determined the one-point function data for the 3d Ising, XY, and Heisenberg models; this thesis will also reproduce these results.

The analytic bootstrap is a method that determines conformal data from the discontinuity of correlators by using the analytic structure as a new constraint. The two primary tools for this approach are the Lorentzian inversion formula [11, 12] and dispersion relations [13], which are known to be equivalent. In the context of finite-temperature CFTs, relevant developments include the inversion formula proposed in [10] and the dispersion relation introduced in [14].

In [1], which we review here, the authors employed this thermal dispersion relation to develop a novel formula for the thermal two-point function. This formula exhibits several advantageous properties: it is convenient to apply, physically sound, and explicitly captures non-perturbative contributions. Following [1], we will derive these formulas for the thermal two-point function. Using this formula, we determine the asymptotic behavior of the thermal OPE coefficients for heavy operators and, subsequently, reproduce the results of [2].

The organization of this thesis is as follows. In Chapter II, we review the necessary conformal field theory background. We begin with the foundations of conformal field theory in dimensions $d > 2$, covering the conformal algebra, radial quantization, and the Operator Product Expansion (OPE). We then introduce the Matsubara formalism for finite-temperature field theory and explain the Kubo-Martin-Schwinger (KMS) condition, which imposes periodicity in imaginary time. Chapter III establishes the framework for CFTs at finite temperature. We discuss how the temperature scale breaks conformal symmetry, leading to non-vanishing thermal one-point functions. We introduce the thermal conformal blocks and explain how the periodic geometry of the thermal cylinder acts as a constraint on the correlation functions. We also briefly discuss Generalized Free Field (GFF) theory as a baseline example. In Chapter IV, we present the main results of the analytic thermal bootstrap. We first analyze the thermal two-point function at zero spatial separation ($x = 0$) and derive an exact formula using dispersion relations. We then extend this method to non-zero spatial separation ($x \neq 0$) by introducing the *generalized method of images*. We apply this technique to the O(N) model. Furthermore, we mathematically prove the asymptotic behavior of heavy OPE coefficients and use this to numerically determine the thermal data for the 3d Ising, XY, and Heisenberg models. Finally, Chapter V summarizes our conclusions and discusses future research directions.

II. PRELIMINARIES

A. Conformal field theory on $\mathbb{R}^{d>2}$

The discussion on conformal field theory presented here combines the mathematical foundations from [15] with the modern bootstrap perspective reviewed in [16].

1. *Conformal transformations*

Let (M, g) and (N, h) be two pseudo-Riemannian manifolds of the same dimension d. A *conformal transformation* is a smooth mapping $\varphi : (M, g) \to (N, h)$ of maximal rank such that the metric is invariant up to a local scale factor $\Omega \in C^\infty(M, \mathbb{R}_{>0})$

$$\varphi^* h = \Omega^2 g, \tag{1}$$

where Ω is called the *conformal factor* of φ. In local coordinates, for $x \in M$,

$$(\varphi^* h)_{\mu\nu}(x) = h_{ij}(\varphi(x))\partial_\mu \varphi^i \partial_\nu \varphi^j, \tag{2}$$

thus,

$$h_{ij}(\varphi(x))\frac{\partial \varphi^i}{\partial x^\mu}(x)\frac{\partial \varphi^j}{\partial x^\nu}(x) = \Omega^2 g_{\mu\nu}(x). \tag{3}$$

In the following, we focus on the case of endomorphisms $\varphi : (M, g) \to (M, g')$ of maximal rank. Denoting $x' = \varphi(x)$, the previous relation can be rewritten as

$$g'_{ij}(x')\frac{\partial x'^i}{\partial x^\mu}\frac{\partial x'^j}{\partial x^\nu} = \Omega^2 g_{\mu\nu}(x). \tag{4}$$

Let $X \in \mathfrak{X}(M)$ be a smooth vector field and let $\Phi_X : M \times \mathbb{R} \to M$, $(x,t) \mapsto \Phi_X^t(x)$ be the flow (one-parameter group) of X. Then X is a *conformal Killing field* if Φ_X^t is a conformal transformation for all t in some neighborhood of 0. To derive the governing equation for X, we differentiate the condition $(\Phi_X^t)^* g = \Omega_t^2 g$ with respect to t at $t=0$. The derivation proceeds as follows:

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \left((\Phi_X^t)^* g\right)_{\mu\nu}(x) &= \left.\frac{d}{dt}\right|_{t=0} g_{ij}\left(\Phi_X^t(x)\right) \frac{\partial(\Phi_X^t)^i}{\partial x^\mu}(x) \frac{\partial(\Phi_X^t)^j}{\partial x^\nu}(x) \\
&= (\partial_k g_{ij})(x) X^k(x) \delta^i_\mu \delta^j_\nu + g_{ij} \partial_\mu X^i(x) \delta^j_\nu + g_{ij} \delta^i_\mu \partial_\nu X^j(x) \\
&= (\partial_k g_{\mu\nu})(x) X^k(x) + g_{\rho\nu}(x) \partial_\mu X^\rho(x) + g_{\mu\rho}(x) \partial_\nu X^\rho(x) \\
&= (\Gamma^\rho{}_{k\mu} g_{\rho\nu} + \Gamma^\rho{}_{k\nu} g_{\mu\rho}) X^k + g_{\rho\nu} \partial_\mu X^\rho + g_{\mu\rho} \partial_\nu X^\rho \\
&= g_{\rho\nu} \nabla_\mu X^\rho + g_{\mu\rho} \nabla_\nu X^\rho \\
&= \nabla_\mu X_\nu + \nabla_\nu X_\mu .
\end{aligned}$$

By defining the *conformal Killing factor* $\kappa(x) := \left.\frac{d}{dt}\right|_{t=0} \Omega_t^2(x)$, we arrive at the conformal Killing equation:

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \kappa g_{\mu\nu}. \tag{5}$$

In this thesis, we restrict our attention to flat Euclidean space $(M,g) = (\mathbb{R}^d, \delta)$ of $d > 2$ dimension. In this setting, the connection coefficients vanish ($\Gamma^\rho{}_{\mu\nu} = 0$), and the conformal Killing equation (5) simplifies to

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \kappa \delta_{\mu\nu}. \tag{6}$$

Tracing (6) with $\delta^{\mu\nu}$,

$$\delta^{\mu\nu}(\partial_\mu X_\nu + \partial_\nu X_\mu) = \kappa \delta_{\mu\nu}\delta^{\mu\nu}, \tag{7a}$$

$$2\partial \cdot X = \kappa d. \tag{7b}$$

In other words, the conformal Killing factor κ is proportional to $(\partial \cdot X)$. Substituting (7b) into (6), we obtain

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \frac{2}{d}(\partial \cdot X)\delta_{\mu\nu}. \tag{8}$$

To determine the higher derivatives of X_μ, we differentiate (8) with respect to ∂^ν:

$$\partial_\mu(\partial \cdot X) + \Box X_\mu = \frac{2}{d}\partial_\mu(\partial \cdot X).$$

Taking the derivative ∂_ν again, we have

$$\partial_\mu\partial_\nu(\partial \cdot X) + \Box\partial_\nu X_\mu = \frac{2}{d}\partial_\mu\partial_\nu(\partial \cdot X). \tag{9}$$

Interchanging $\mu \leftrightarrow \nu$,

$$\partial_\nu\partial_\mu(\partial \cdot X) + \Box\partial_\mu X_\nu = \frac{2}{d}\partial_\nu\partial_\mu(\partial \cdot X). \tag{10}$$

By adding (9) and (10) and simplifying, we find

$$(\delta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)(\partial \cdot X) = 0 \quad \forall \mu,\nu. \tag{11}$$

Therefore, $\Box(\partial \cdot X) = 0$, which implies $\partial_\mu\partial_\nu(\partial \cdot X) = 0$ for all μ,ν.

2. *Classification of conformal transformations in $d > 2$*

For $d > 2$, (11) constrains the terms of order 2 or higher in $\partial \cdot X$ to vanish. This means that the solutions $\partial \cdot X$ of the conformal Killing equation (11) are the affine maps

$$(\partial \cdot X)(x) = \lambda + \alpha_\mu x^\mu, \quad x \in M, \tag{12}$$

with $\lambda, \alpha_\mu \in \mathbb{R}$.

We first determine the conformal Killing fields X with conformal Killing factor $\kappa = 0$. $\partial_\mu X_\nu + \partial_\nu X_\mu = 0$ implies that X^μ can be written as

$$X^\mu(x) = c^\mu + \omega^\mu{}_\nu x^\nu, \tag{13}$$

where $c^\mu \in \mathbb{R}$, $\omega \in \mathfrak{so}(d) := \left\{\omega \,|\, \omega^\mathsf{T} + \omega = 0\right\}$. This vector field defines the flow $\Phi_X^t(x)$ through the differential equation

$$\frac{\partial}{\partial t}\left(\Phi_X^t(x)\right)^\mu = c^\mu + \omega^\mu{}_\nu\left(\Phi_X^t(x)\right)^\nu. \tag{14}$$

Here, the constant term c^μ corresponds to a *translation*, yielding $\Phi_X^1(x) = x + c$ (when $\omega = 0$), while the linear term involving ω generates a *rotation*, resulting in $\Phi_X^1(x) = \Lambda x$, with $\Lambda = e^\omega \in SO(d)$ (when $c = 0$).

Next, we consider the case of a constant non-vanishing conformal factor $\kappa = \lambda \in \mathbb{R} \setminus \{0\}$. This corresponds to the generator of *dilatations*, given by the vector field

$$X^\mu(x) = \lambda x^\mu. \tag{15}$$

The flow equation $\frac{\partial}{\partial t}\Phi^t_X(x) = \lambda\Phi^t_X(x)$ admits the solution $\Phi^t_X(x) = e^{\lambda t}x$. Thus, for unit time $t = 1$, this generates the scaling transformation $\Phi^1_X(x) = e^{\lambda}x$.

Finally, we examine the *special conformal transformations* (SCT), characterized by a position-dependent conformal factor $\kappa(x) = 4b^\mu x_\mu$ with a parameter vector $b \in \mathbb{R}^d$. The corresponding Killing field is given by

$$X^\mu(x) = 2(b \cdot x)x^\mu - x^2 b^\mu. \tag{16}$$

The flow equation $\frac{\partial}{\partial t}\Phi^t_X(x) = X(\Phi^t_X(x))$ involves a quadratic dependence on coordinates. The solution at $t = 1$ yields the standard fractional linear transformation form:

$$\Phi^1_X(x)^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2}. \tag{17}$$

Here, strictly speaking, the vector field X is not complete on $\mathbb{R}^d$ due to the singularity in the denominator. However, it generates a global flow on the conformally compactified space $S^d \cong \mathbb{R}^d \cup \{\infty\}$. Here, we consider the finite transformation generically for points where the denominator implies no singularity.

3. *Conformal algebra* $\mathfrak{so}(d+1,1)$

Consequently, the vector fields determined by the conformal Killing equation constitute the generators of the conformal group. Acting on the space of scalar functions (fields), these generators can be represented as differential operators:

$$P_\mu = \partial_\mu, \tag{18a}$$
$$M_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu, \tag{18b}$$
$$D = x^\mu\partial_\mu, \tag{18c}$$
$$K_\mu = 2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu. \tag{18d}$$

Here, P_μ generates translations, $M_{\mu\nu}$ generates rotations (generating the $\mathfrak{so}(d)$ subalgebra), D generates dilatations, and K_μ generates special conformal transformations. The total number of generators is $\frac{(d+2)(d+1)}{2} = d + \frac{d(d-1)}{2} + 1 + d$.

These operators form a closed Lie algebra, known as the *conformal algebra*. The non-vanishing commutation relations are:

- Usual Poincaré algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\rho\mu} - \delta_{\mu\sigma}M_{\rho\nu}, \tag{19a}$$
$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu. \tag{19b}$$

- New conformal relations:

$$[D, P_\mu] = P_\mu, \tag{20a}$$
$$[D, K_\mu] = -K_\mu, \tag{20b}$$
$$[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu} \tag{20c}$$
$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu \tag{20d}$$

To understand the structure of the conformal algebra, let us define the following generators:

$$L_{\mu\nu} = M_{\mu\nu}, \tag{21a}$$
$$L_{-1,0} = D, \tag{21b}$$
$$L_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu), \tag{21c}$$
$$L_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu), \tag{21d}$$

with the antisymmetry condition $L_{ab} = -L_{ba}$ for indices $a, b \in \{-1, 0, 1, \cdots, d\}$. Using the standard commutation relations of the conformal group, one can verify that these generators satisfy the following algebra:

$$\begin{aligned}
[L_{-1,0}, L_{0,\mu}] &= L_{-1,\mu},\\
[L_{-1,0}, L_{-1,\mu}] &= L_{0,\mu},\\
[L_{0,\mu}, L_{0,\nu}] &= -L_{\mu\nu},\\
[L_{-1,\mu}, L_{-1,\nu}] &= L_{\mu\nu},\\
[L_{0,\mu}, L_{-1,\nu}] &= -\delta_{\mu\nu}L_{-1,0} + L_{\mu\nu}.
\end{aligned} \tag{22}$$

These relations (22) can be summarized in a unified form by introducing a metric tensor $\eta_{ab} = \mathrm{diag}(-1, 1, 1, \ldots, 1)$, where $\eta_{-1,-1} = -1$, $\eta_{00} = 1$, and $\eta_{\mu\nu} = \delta_{\mu\nu}$. The conformal algebra is then isomorphic to the orthogonal Lie algebra $\mathfrak{so}(d+1,1)$, expressed as:

$$[L_{ab}, L_{cd}] = \eta_{bc}L_{ad} + \eta_{ad}L_{bc} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac}. \tag{23}$$

Since the complexification of $\mathfrak{so}(d+1,1)$ is $(\mathfrak{so}(d+1,1))_{\mathbb{C}} \cong \mathfrak{so}(d+2,\mathbb{C})$ via $\mathrm{diag}(1, 1, \ldots, 1, i)$, the structure of the conformal algebra can be understood through the structure of the orthogonal Lie algebra $\mathfrak{so}(d+2)$. Given that $\mathfrak{so}(d+2)$ is of rank $\lfloor \frac{d+2}{2} \rfloor = \lfloor \frac{d}{2} \rfloor + 1$, we need to identify $\lfloor \frac{d}{2} \rfloor + 1$ basis elements of the Cartan subalgebra. Following standard convention, we first choose $\lfloor \frac{d}{2} \rfloor$ generators of rotations $M_{12}, M_{34}, \cdots$ in orthogonal planes. As the dilatations generator D commutes with these rotations, it is added to form the full Cartan subalgebra $\mathrm{span}(D, M_{12}, M_{34}, \cdots)$ of the conformal algebra. This choice of the Cartan subalgebra is physically realized in the *radial quantization* scheme (see Section II A 4), where the dilatation operator D plays the role of the Hamiltonian generating 'time' ($r = e^\tau$) translations along the radial direction, and the rotations M_{ij} act on the constant-radius spheres.

Consequently, the states (or the representation of $\mathfrak{so}(d+1,1)$) are labeled by the weights of this Cartan subalgebra, corresponding to the *scaling dimension* Δ and the *spins*. The commutation relations (20a) and (20b) establish that P_μ and K_μ act as ladder operators that shift the scaling dimension. Furthermore, their appropriate complex linear combinations (e.g., $P_1 \pm iP_2$ and

$K_1 \pm iK_2$) serve as ladder operators for the spins defined by the rotation generators.

4. Conformal field theories in $d > 2$ dimensions

In conformal field theories, because asymptotic particles cannot be isolated, we cannot define the theory by constructing non-interacting in- and out-states, unlike in standard quantum field theory. Instead, to construct the Hilbert space, one uses *radial quantization* on the geometry $\mathbb{R} \times S^{d-1}$, where the distance from the origin is regarded as time $\tau = \ln r$ with radial ordering $\langle \mathcal{R}\{\cdots\}\rangle$. The sphere S^{d-1} has a corresponding Hilbert space $\mathcal{H}$ which carries a unitary representation of the conformal algebra $\mathfrak{so}(d+1,1)$, and the action of a symmetry generator Q is defined as inserting the surface operator $Q(S^{d-1})$ on the sphere.

In radial quantization, a state on the sphere S^{d-1} at $\tau = 0$ (equivalently $r = 1$) is prepared by performing the path integral over the interior B of the sphere. The vacuum $|0\rangle$ corresponds to the path integral with no operator insertions. Moreover, the state $\mathcal{O}(x)|0\rangle$ is defined by the path integral over B with an insertion of $\mathcal{O}(x)$ at the point $x \in B$:

$$\langle \phi_b|\mathcal{O}(x)|0\rangle = \int_{\substack{\varphi|_{\partial B}=\phi_b \\ r\leq 1}} \mathcal{D}\varphi e^{-S[\varphi]}\mathcal{O}(x), \tag{24}$$

where ϕ_b is a field configuration on the sphere ∂B. This is physically equivalent to preparing an in-state at past infinity $\tau \to -\infty$ on the cylinder $\mathbb{R} \times S^{d-1}$. This construction of the state provides a map from operators to states.

Conversely, given a state, we can construct the corresponding local operator. Following the discussion in Section II A 3, we decompose $\mathcal{H}$ into a direct sum of irreducible representations of the conformal algebra:

$$\mathcal{H} = \bigoplus_i \mathcal{V}_{\Delta_i,\ell_i} \tag{25}$$

Here, the index i labels the primary states $|\Delta_i, \ell_i\rangle$, specified by their scaling dimension Δ_i and their spin $\ell_i = (\ell_i^1, \cdots, \ell_i^{\lfloor d/2\rfloor})$. Each module $\mathcal{V}_{\Delta_i,\ell_i}$ is the *conformal family* obtained by acting with the momentum generators P^μ on the primary (*lowest*-weight of D) state and then quotienting by null states, which removes any null states. This leads to the conclusion that an arbitrary state in $\mathcal{H}$ can be written as a linear combination of simultaneous eigenstates of D and $M_{\mu\nu}$. Denoting such an eigenstate by $|\Delta, \ell\rangle$, the actions of D and $M_{\mu\nu}$ on $|\Delta, \ell\rangle$ are given by

$$D|\Delta, \ell\rangle = \Delta|\Delta, \ell\rangle, \tag{26a}$$

$$M_{\mu\nu}|\Delta, \ell\rangle^a = (\mathcal{S}_{\mu\nu})_b{}^a|\Delta, \ell\rangle^b, \tag{26b}$$

where $\mathcal{S}_{\mu\nu}$ are matrices satisfying the same algebra as $M_{\mu\nu}$, and a, b are indices for the $\mathfrak{so}(d)$ representation.

Let $\{|\mathcal{O}_i\rangle\}_i$ be eigenstates of the dilatation (they need not be primary)

$$D|\mathcal{O}_i\rangle = \Delta_i|\mathcal{O}_i\rangle. \tag{27}$$

Using these states, we can construct the corresponding operators $\mathcal{O}_i$ which generate $|\mathcal{O}_i\rangle$. Schematically, a local operator $\mathcal{O}_i(x_i)$ at x_i is defined by cutting the path integral region into the exterior and the interior of a ball B_i of radius ϵ centered at x_i, and then gluing the state $|\mathcal{O}_i\rangle$ onto the boundary ∂B_i to replace the interior integration. This procedure gives a quantity that behaves exactly like a correlator of local operators:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \int \prod_{i=1}^n \mathcal{D}\phi_{b_i}\langle\phi_{b_i}|\mathcal{O}_i\rangle \int_{\substack{\varphi|_{\partial B_i}=\phi_{b_i} \\ r>\epsilon}} \mathcal{D}\varphi e^{-S[\varphi]}, \tag{28}$$

where the integrals $\mathcal{D}\phi_{b_i}$ are performed over field configurations on the boundaries ∂B_i.

By conformal symmetry, the cutoff ϵ is not physical. Define $\mathcal{O}_i(x_i;\epsilon)$ as the insertion obtained by the above procedure. A dilatation about x_i, $r_i \mapsto \lambda r_i$, maps $\epsilon \mapsto \lambda\epsilon$. By (27), we have $\mathcal{O}_i(x_i;\lambda\epsilon) = \lambda^{-\Delta_i}\mathcal{O}_i(x_i;\epsilon)$, so $(\lambda\epsilon)^{\Delta_i}\mathcal{O}_i(x_i;\lambda\epsilon) = \epsilon^{\Delta_i}\mathcal{O}_i(x_i;\epsilon)$. Thus we can define the ϵ-independent local operator by

$$\mathcal{O}_i(x_i) := \lim_{\epsilon\to 0}\epsilon^{\Delta_i}\mathcal{O}_i(x_i;\epsilon). \tag{29}$$

From these two constructions, we conclude that there is a one-to-one correspondence between states on the sphere and local operators inserted at its center. This is the "state-operator correspondence." In other words, a local operator $\mathcal{O}(0)$ is identified with the state $\mathcal{O}(0)|0\rangle := |\mathcal{O}\rangle$. We can now define a *conformal field theory.* First, a primary operator $\mathcal{O}$ creates a state that is killed by K_μ, i.e.

$$[K_\mu, \mathcal{O}(0)] = 0 \;\longleftrightarrow\; K_\mu|\mathcal{O}\rangle = 0, \tag{30a}$$

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0) \;\longleftrightarrow\; D|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle, \tag{30b}$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = S_{\mu\nu}\mathcal{O}(0) \;\longleftrightarrow\; M_{\mu\nu}|\mathcal{O}\rangle = S_{\mu\nu}|\mathcal{O}\rangle. \tag{30c}$$

We now define the *conformal family* $[\mathcal{O}]$. It is the subspace of $\mathcal{H}$ spanned by the primary state $|\mathcal{O}\rangle$ and its descendants, obtained by acting with P_μ on $|\mathcal{O}\rangle$. A general descendant state is given by

$$P_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}|\mathcal{O}\rangle. \tag{31}$$

In terms of operators, this is identified with the derivatives taken at the origin:

$$\partial_\mu\mathcal{O}(x)|_{x=0}|0\rangle = [P_\mu, \mathcal{O}(0)]|0\rangle = P_\mu|\mathcal{O}\rangle. \tag{32}$$

The state $\mathcal{O}(x)|0\rangle$ can be expanded as a (infinite) superposition of descendants:

$$\mathcal{O}(x)|0\rangle = e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}|0\rangle = e^{x\cdot P}|\mathcal{O}\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}(x\cdot P)^n|\mathcal{O}\rangle. \quad (33)$$

A conformal field theory in $d > 2$ dimensions ($\mathrm{CFT}_{d>2}$) consists of:

- a unique vacuum state $|0\rangle$ annihilated by the generators of the conformal algebra,
- a collection of conformal families $[\mathcal{O}_i]$.

5. *Correlation functions*

Conformal symmetry completely determines the form of low-point correlation functions of primary operators, up to structure constants.

First, consider the one-point function $\langle\phi(x)\rangle$ of a primary operator. Translational invariance requires the correlation functions to be independent of the position x. Furthermore, under a dilatation $x \mapsto \lambda x$, the operator transforms as $\phi(x) \mapsto \phi'(\lambda x) = \lambda^{-\Delta}\phi(x)$. Since the vacuum $|0\rangle$ is invariant under the conformal transformation, the one-point function must satisfy the scaling relation $\langle\phi(x)\rangle = \lambda^{\Delta}\langle\phi(\lambda x)\rangle$. Combined with translational invariance, this implies $\langle\phi(x)\rangle = \lambda^{\Delta}\langle\phi(x)\rangle$, which necessitates that the one-point function vanishes for any operator with a non-zero scaling dimension:

$$\langle\phi(x)\rangle = \begin{cases} 1 & \text{if } \phi = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \quad (34)$$

Next, we examine the two-point function of two scalar primary fields, $\langle\phi_1(x_1)\phi_2(x_2)\rangle$. Poincaré symmetry restricts the two-point function to depend only on the distance $|x_1 - x_2|$, i.e., $\langle\phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|)$. Scale invariance constrains the two-point function to satisfy

$$\begin{aligned} 0 &= \langle 0|[D, \phi_1(x_1)\phi_2(x_2)]|0\rangle \\ &= \langle 0|[D, \phi_1(x_1)]\phi_2(x_2) + \phi_1(x_1)[D, \phi_2(x_2)]|0\rangle \\ &= (x_1^\mu\partial_\mu^1 + \Delta_1 + x_2^\mu\partial_\mu^2 + \Delta_2)\langle\phi_1(x_1)\phi_2(x_2)\rangle \end{aligned} \quad (35)$$

This gives the power-law dependence $f(|x_1 - x_2|) \propto |x_1 - x_2|^{-(\Delta_1+\Delta_2)}$. Invariance under special conformal transformations further imposes the selection rule that the correlation function vanishes unless $\Delta_1 = \Delta_2$.To demonstrate this, consider the correlator with one operator shifted to the origin, $g(x) = \langle\phi_1(0)\phi_2(x)\rangle$. Since the vacuum is invariant under the SCT generator K_μ, and a primary operator at the origin satisfies $[K_\mu, \phi_1(0)] = 0$, the invariance condition implies:

$$0 = \langle 0|[K_\mu, \phi_1(0)\phi_2(x)]|0\rangle = \langle\phi_1(0)[K_\mu, \phi_2(x)]\rangle. \quad (36)$$

Substituting the differential action of K_μ on $\phi_2(x)$, given by $[K_\mu, \phi_2(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu - 2x_\mu\Delta_2)\phi_2(x)$, into the scale-invariant form $G(x) \propto |x|^{-(\Delta_1+\Delta_2)}$, we obtains the constraint:

$$(\Delta_1 - \Delta_2)x_\mu g(x) = 0. \quad (37)$$

Thus, for a non-vanishing correlator, the scaling dimensions must be identical. The resulting form is

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \text{if } \Delta_1 = \Delta_2, \\ 0 & \text{if } \Delta_1 \neq \Delta_2, \end{cases} \quad (38)$$

where C_{12} is a constant.

6. *The operator product expansion*

The Operator Product Expansion (OPE) states that the product of two operators can be expanded as:

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2), \quad (39)$$

where k runs over primaries and C_{ijk} is a differential operator containing the OPE coefficients $f_{ijk} \in \mathbb{R}$ of the form

$$\begin{aligned} C_{ijk}(x, \partial) =& \frac{f_{ijk}}{|x|^{\Delta_i+\Delta_j-\Delta_k}}(1 + \#x^\mu\partial_\mu \\ &+ \#x^\mu x^\nu\partial_\mu\partial_\nu + \#x^2\partial^2 + \cdots). \end{aligned} \quad (40)$$

The derivation begins with the fact that the state created by the product is a linear combination of primaries and descendants:

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_k C_{ijk}(x, P)\mathcal{O}_k(0)|0\rangle. \quad (41)$$

By performing radial quantization centered at the origin, this state decomposition (41) translates into the operator expansion given above.

The OPE is of central importance in conformal field theory for two main reasons. First, unlike in general quantum field theories where it is an asymptotic approximation, the OPE in a CFT is an exact statement with a finite radius of convergence (limited only by the distance to the nearest other operator insertion). This allows it to be used at finite operator separations. Second, it provides a powerful algorithm to compute higher-point correlation functions. By recursively applying the expansion inside a correlator, any n-point function can be reduced to a linear combination of $(n-1)$-point functions, and ultimately to known one-point and two-point functions. Consequently, a CFT is completely defined by its "dynamical data" called the *conformal data*: the spectrum of scaling dimensions $\{\Delta_\mathcal{O}\}$, spins $\{J_\mathcal{O}\}$, and the OPE constants $\{f_{\mathcal{O}\mathcal{O}'\mathcal{O}''}\}$. The consistency of this expansion, which corresponds to the associativity of the operator algebra, imposes strong constraints (the *cross-*

ing symmetry equations) on the conformal data, and the strategy of exploiting these constraints is known as the *conformal bootstrap.*

7. *Twist and double-twist operators*

The *twist* τ of an operator $\mathcal{O}$ with dimension Δ and spin ℓ is defined by $\tau = \Delta - \ell$.

Theorem 1 (Existence of double-twist operators [17–19]) *Suppose a CFT in $d > 2$ dimensions contains primary operators $\mathcal{O}_1$, $\mathcal{O}_2$ with twists τ_1, τ_2. For each $n = 0, 1, 2, \cdots$, there exists an infinite family of primary operators with increasing spin and twists approaching $\tau_1 + \tau_2 + 2n$ as $\ell \to \infty$.*

Schematically, these operators are

$$\mathcal{O}_1 \partial^{\mu_1} \cdots \partial^{\mu_\ell} \partial^{2n} \mathcal{O}_2.$$

We denote the family with twist approaching $\tau_1 + \tau_2 + 2n$ as $[\mathcal{O}_1 \mathcal{O}_2]_n$ and refer to such operators as "double-twist" operators.

Physically, these operators represent two particles orbiting each other with high angular momentum. In the large spin limit $\ell \to \infty$, the constituents are effectively separated and non-interacting, so the anomalous dimension vanishes. At finite spin, however, interactions induce corrections

$$\Delta_{n,\ell} = \tau_1 + \tau_2 + 2n + \ell + O(1/\ell). \tag{42}$$

In this thesis, we neglect the anomalous dimensions of double-twist operators, as they are suppressed by $1/\ell$. Double-twist operators play a central role in the conformal bootstrap, as they enable analytic solutions to the crossing equations and dominate the heavy operator spectrum in bootstrap computations. In many of the computations presented here, we proceed under the assumption that the heavy operator spectrum consists only of these double-twist families.

B. Matsubara formalism and KMS relation

The conformal field theory discussed so far was the CFT at zero temperature without thermal effects. To study the finite temperature effects of quantum field theory, we must first define the thermal mixed state and identify the thermal expectation value with the correlation function defined by the path-integral formulation. Specifically, for the thermal equilibrium state, we can describe observables through the *Matsubara*, or *imaginary time*, *formalism* [20]. Under the Matsubara formalism, one can prove that the thermal field theory in Minkowski space $\mathbb{R}^{1,3}$ corresponds to the field theory on a cylinder $S^1_\beta \times \mathbb{R}^{d-1}$ with Euclidean signature where the time direction is compactified. That is, the thermal effects in quantum field theory are completely encoded into the geometry. This periodic boundary condition gives additional constraints that enable the conformal bootstrap at finite temperature. Based on Chapter 1 of [21], we briefly review thermal field theory in this section.

1. *Matsubara formalism*

Here we assume that our state is in thermal equilibrium. The density matrix ρ_β corresponding to an equilibrium state at $\beta = 1/T$ (we assume that the Boltzmann constant $k_B = 1$) is

$$\rho_\beta = \frac{1}{\mathcal{Z}} e^{-\beta \mathcal{H}}, \tag{43}$$

where $\mathcal{H}$ is the Hamiltonian of the system and $\mathcal{Z}$ is the partition function of the system defined as

$$\mathcal{Z} = \operatorname{Tr} e^{-\beta \mathcal{H}}. \tag{44}$$

Thermal expectation values are defined by

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_\beta = \frac{1}{\mathcal{Z}} \operatorname{Tr} \left(e^{-\beta \mathcal{H}} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \right), \tag{45}$$

where we use the notation $\langle \cdots \rangle_\beta$ to denote thermal expectation values.

The Matsubara formalism is an approach to calculate thermal observables by employing imaginary time $t = -i\tau \in i\mathbb{R}$ with τ-ordering $\langle P_\tau \{ \cdots \} \rangle_\beta$, so that

$$\mathcal{O}(\tau) = e^{\tau \mathcal{H}} \mathcal{O} e^{-\tau \mathcal{H}}. \tag{46}$$

To connect thermal expectation values with correlation functions, we write the trace as an functional integral over all possible field configurations ϕ_0:

$$\mathcal{Z} = \int \mathcal{D}\phi_0 \langle \phi_0 | e^{-\beta \mathcal{H}} | \phi_0 \rangle. \tag{47}$$

Here, $|\phi_0\rangle$ represents the state of the field at the initial time $\tau = 0$. Because of the trace, the final state at $\tau = \beta$ must be the same state $|\phi_0\rangle$. Then, we split the imaginary time interval β into N small slices of duration $\epsilon = \beta/N$:

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = \beta, \tag{48}$$

and for each time τ_i, denote ϕ_i the intermediate fields. For $\epsilon \to 0$, we can rewrite the time evolution operator $e^{-\beta \mathcal{H}}$ as

$$e^{-\beta \mathcal{H}} = e^{-\epsilon \mathcal{H}} e^{-\epsilon \mathcal{H}} \cdots e^{-\epsilon \mathcal{H}}. \tag{49}$$

Substituting (49) into (47) and using the completeness relation, we obtain

$$\mathcal{Z} = \int \prod_{k=0}^{N-1} \mathcal{D}\phi_k \langle \phi_{k+1} | e^{-\epsilon \mathcal{H}} | \phi_k \rangle, \tag{50}$$

with the periodic boundary condition $\phi_N = \phi_0$. Let π_i be the conjugate momentum of ϕ_i. Then a single transition amplitude between time slice k and $k+1$ is

$$\begin{aligned}\langle \phi_{k+1}|e^{-\epsilon\mathcal{H}}|\phi_k\rangle \\ &= \int \frac{\mathcal{D}\pi_k}{2\pi} \exp\left(\int d^3x\,(i\pi_k(\phi_{k+1}-\phi_k)-\epsilon\mathcal{H})\right) \\ &\overset{\epsilon\to 0}{=\!=} \int \frac{\mathcal{D}\pi_k}{2\pi} \exp\left(\epsilon\int d^3x\left(i\pi_k\dot{\phi}_k - \mathcal{H}\right)\right).\end{aligned} \tag{51}$$

By taking the continuum limit, we multiply all N slices together such that the sum over slices becomes an integral over the imaginary time τ ($\sum_k \epsilon \mapsto \int_0^\beta d\tau$). This defines the phase space path integral, where we sum over all possible temporal paths of both the fields and their conjugate momenta:

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}\pi \exp\left(\int_0^\beta d\tau \int d^3x \left[i\pi\dot{\phi} - \mathcal{H}\right]\right). \tag{52}$$

Consequently, by introducing the Euclidean action $S = \int_0^\beta d\tau \left[\mathcal{H} - i\pi\dot{\phi}\right]$, the partition function can be written in the following standard form:

$$\mathcal{Z} = \operatorname{Tr} e^{-\beta\mathcal{H}} \overset{!}{=} \int \mathcal{D}\phi e^{-S[\phi]}. \tag{53}$$

By the same logic, one can find that thermal expectation values correspond to correlation functions defined through path integrals:

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_\beta \\ &= \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-S[\phi]}.\end{aligned} \tag{54}$$

2. KMS relation and (anti-)periodicity

The Kubo-Martin-Schwinger (KMS) relation [22, 23] states that, for a thermal two-point function,

$$\langle A(t)B(0)\rangle_\beta = \langle B(0)A(t+i\beta)\rangle_\beta, \tag{55}$$

or, in Euclidean time $\tau = it$,

$$\langle A(\tau)B(0)\rangle_\beta = \langle B(0)A(\tau-\beta)\rangle_\beta. \tag{56}$$

This can be proven by a straightforward calculation as follows:

$$\begin{aligned}\langle A(\tau)B(0)\rangle_\beta &= \mathcal{Z}^{-1}\operatorname{Tr} e^{-\beta\mathbf{H}}A(\tau)B(0) \\ &= \mathcal{Z}^{-1}\operatorname{Tr} e^{-\beta\mathbf{H}}e^{\tau\mathbf{H}}A(0)e^{-\tau\mathbf{H}}B(0) \\ &= \mathcal{Z}^{-1}\operatorname{Tr} e^{(\tau-\beta)\mathbf{H}}A(0)e^{-(\tau-\beta)\mathbf{H}}e^{-\beta\mathbf{H}}B(0) \\ &= \mathcal{Z}^{-1}\operatorname{Tr} e^{-\beta\mathbf{H}}B(0)A(\tau-\beta) = \langle B(0)A(\tau-\beta)\rangle_\beta.\end{aligned} \tag{57}$$

We define the τ-ordering operator P_τ as follows:

$$P_\tau A(\tau)B(\tau') := \theta(\tau-\tau')A(\tau)B(\tau') \pm \theta(\tau'-\tau)B(\tau')A(\tau), \tag{58}$$

where the $-$ sign in the second term is for fermionic operators. From this point on, we assume that the thermal expectation value $\langle\cdots\rangle_\beta$ is defined with this P_τ ordering. Combining this definition (58) with the KMS condition leads to the following (anti-)periodicity property:

$$\begin{aligned}\langle A(\tau)B(0)\rangle_\beta &= \pm\langle B(0)A(\tau)\rangle_\beta \\ &= \pm\langle A(\tau+\beta)B(0)\rangle_\beta.\end{aligned} \tag{59}$$

Here, $+$ sign for bosonic operators and $-$ sign for fermionic operators. Since we focus on bosonic operators in this thesis, the relation (59) simplifies to

$$\langle A(\tau)B(0)\rangle_\beta = \langle A(\tau+\beta)B(0)\rangle_\beta. \tag{60}$$

By the same reasoning, all bosonic correlation functions are periodic. This is equivalent to compactifying the time direction on a circle S^1_β.

III. CONFORMAL FIELD THEORY AT FINITE TEMPERATURE

A. Thermal conformal data and OPE

In conformal field theory at finite temperature, as the length of the time direction takes on physical meaning, the scaling symmetry of the time direction becomes broken. Due to this symmetry breaking, the one-point function no longer vanishes. That is, at finite temperature, the conformal data introduced in section II A 6 alone cannot characterize the CFT.

To find the thermal one-point function, let us determine the symmetry of the $S^1_\beta \times \mathbb{R}^{d-1}$ geometry. Firstly, $S^1_\beta \times \mathbb{R}^{d-1}$ is clearly invariant under the conformal transformation of $\mathbb{R}^{d-1}$. Additionally, it is also invariant under $[\tau \leftrightarrow -\tau]\times[x^1 \leftrightarrow -x^1]$, which is a transformation rotating the cylinder $S^1_\beta \times \mathbb{R}^{d-1}$ by 180 degrees. That is, the symmetry group includes $O(d-1)$. By this symmetry, we can determine the thermal one-point function.

Let us consider a primary operator $\mathcal{O}$ with dimension Δ and spin J. By the first fundamental theorem of invariant theory, all invariant tensors must be generated only by $e^\mu = (1, 0, \cdots, 0)$ and $g^{\mu\nu} = \delta^{\mu\nu}$. That is, $\mathcal{O}^{\mu_1\cdots\mu_J}$ is either a scalar or a symmetric traceless tensor. Additionally, by the $\tau \leftrightarrow -\tau$ symmetry, the spin is even, $J \in 2\mathbb{Z}_{\geq 0}$. On the other hand, by dimensional analysis, it must be $\langle\mathcal{O}(0)\rangle_\beta \propto \beta^{-\Delta}$. These three facts imply that the thermal one-point function must be written in the following form:

$$\langle \mathcal{O}^{\mu_1\cdots\mu_J}(\tau,\mathbf{x})\rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^\Delta}(e^{\mu_1}\cdots e^{\mu_J} - \text{traces}), \tag{61}$$

where $b_{\mathcal{O}}$ is a dimensionless one-point function coefficient, which is the new data introduced by finite temperature effects. Here, $-$traces means that terms corresponding to the trace of $e^{\mu_1} \cdots e^{\mu_J}$ have been subtracted so that this tensor becomes traceless. For example, when $J = 2, 3, 4$, the symmetric traceless tensors are, respectively,

$$\begin{aligned} & e^{\mu_1} e^{\mu_2} - \frac{1}{d} \delta^{\mu_1 \mu_2}, \\ & e^{\mu_1} e^{\mu_2} e^{\mu_3} - \frac{1}{d+2} \left(e^{\mu_1} \delta^{\mu_2 \mu_3} + e^{\mu_2} \delta^{\mu_1 \mu_3} + e^{\mu_3} \delta^{\mu_1 \mu_2} \right), \\ & e^{\mu_1} e^{\mu_2} e^{\mu_3} e^{\mu_4} \\ & \qquad - \frac{1}{d+4} \left(e^{\mu_1} e^{\mu_2} \delta^{\mu_3 \mu_4} + \text{perm.} \right) \\ & \qquad + \frac{1}{(d+4)(d+2)} \left(\delta^{\mu_1 \mu_2} \delta^{\mu_3 \mu_4} + \text{perm.} \right). \end{aligned}$$

The complete thermal conformal data is

$$\text{thermal conformal data} = \{ \Delta_{\mathcal{O}}, J_{\mathcal{O}}, b_{\mathcal{O}}, f_{\mathcal{O}\mathcal{O}'\mathcal{O}''} \}. \quad (62)$$

This follows from the fact that the OPE converges in a finite region because $S^1_\beta \times \mathbb{R}^{d-1}$ is conformally flat. However, unlike zero-temperature CFTs, the OPE does not converge on the whole space. Indeed, a crucial feature of thermal CFTs is the limited convergence radius of the OPE. This is related to the validity of radial quantization. The OPE requires us to insert a complete set of states on a sphere surrounding the operator at the origin. For the expansion to converge, this sphere must not enclose any other operator insertions. However, due to the periodic nature of the time direction, an operator inserted at the origin has periodic images at distances $n\beta$ (where n is an integer). The closest singularity arises from the operator's own image at $\tau = \beta$. This means that we cannot define a quantization sphere with a radius larger than β without hitting this singularity. Consequently, the radius of convergence for the OPE in thermal CFTs is strictly limited to β.

Explicitly, we can use the OPE to obtain the two-point function $g(\tau, x := |\mathbf{x}|)$:

$$g(\tau, x) = \sum_{\mathcal{O} \in \phi \times \phi} a_{\mathcal{O}} f_{\Delta, J}(\tau, x), \quad (63)$$

where the sum runs over the operators $\mathcal{O}$ (of dimension Δ, spin J) exchanged in the OPE $\phi \times \phi$. Here, the *thermal conformal blocks* $f_{\Delta, J}$ is given by

$$f_{\Delta, J}(\tau, x) = |\tau^2 + x^2|^{\frac{\Delta - 2\Delta_\phi}{2}} C_J^{(\nu)} \left(\frac{\tau}{\sqrt{\tau^2 + x^2}} \right),$$

where $C_J^{(\nu)}(x)$ is the Gegenbauer polynomial and $\nu = (d-2)/2$. The *thermal OPE coefficient* $a_{\mathcal{O}}$ is defined as

$$a_{\mathcal{O}} = \frac{J!}{2^J (\nu)_J} \frac{f_{\phi\phi\mathcal{O}} b_{\mathcal{O}}}{c_{\mathcal{O}}},$$

where $(\nu)_J = \Gamma(\nu + J)/\Gamma(\nu)$, $f_{\phi\phi\mathcal{O}}$ is the zero-temperature three-point function OPE coefficient, and $c_{\mathcal{O}}$ is the normalization of the zero-temperature two-point function $\langle \mathcal{O}\mathcal{O} \rangle$. In this situation, the OPE converges in

$$\sqrt{\tau^2 + x^2} < \beta. \quad (64)$$

We define the complex coordinates as $z = \tau + ix$ and $\bar{z} = \tau - ix$. For notational simplicity, we denote $g(\tau, x)$ as $g(z, \bar{z})$. Additionally, it is also useful to introduce the variables r and w via the relations:

$$r^2 = z\bar{z}, \quad w^2 = \frac{z}{\bar{z}}. \quad (65)$$

Then the conformal block can be written as

$$f_{\Delta, J}(z, \bar{z}) = (z\bar{z})^{\frac{\Delta - 2\Delta_\phi}{2}} C_J^{(\nu)} \left(\frac{z + \bar{z}}{2\sqrt{z\bar{z}}} \right). \quad (66)$$

	Zero-temperature CFT	Thermal CFT
Two-point function	$\frac{C}{\sqrt{\tau^2+x^2}^{2\Delta_\phi}}$	$g(\tau, x)$
OPE	converges everywhere	$\sqrt{\tau^2 + x^2} < \beta$
Conformal data	$\Delta_{\mathcal{O}},\ J_{\mathcal{O}},\ f_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}$	$b_{\mathcal{O}}$ (new unknown)

TABLE I. Comparison between Zero-temperature and Thermal CFT.

At zero spatial distance $x = 0$, the OPE becomes

$$g(\tau) := g(\tau, 0) = \frac{1}{\tau^{2\Delta_\phi}} \sum_{\Delta} a_\Delta \left(\frac{\tau}{\beta} \right)^{\Delta}. \quad (67)$$

Here we defined the reduced OPE coefficients

$$a_\Delta = \sum_{\substack{\mathcal{O} \in \phi \times \phi, \\ \Delta_{\mathcal{O}} = \Delta}} a_{\mathcal{O}} C_J^{(\nu)}(1). \quad (68)$$

B. Periodicity as a bootstrap constraint

The reason we can perform the conformal bootstrap even though scaling symmetry in the time direction is broken is because the $S^1_\beta \times \mathbb{R}^{d-1}$ geometry gives strong constraints on the thermal conformal data. This is because the OPE is not periodic, whereas $g(\tau, x)$ is periodic.

For simplicity, from now on, let us set $\beta = 1$ without loss of generality. Then the KMS condition with the $[\tau \leftrightarrow -\tau]$ symmetry imposes

$$g(\tau, x) = g(1 - \tau, x), \quad (69a)$$
$$g(z, \bar{z}) = g(1 - z, 1 - \bar{z}). \quad (69b)$$

We call *s-channel* the left-hand side and *t-channel* the right-hand one, by analogy to crossing symmetry at zero temperature. Since the correlators are real, it follows that

$$g(z,\bar{z}) = g(\bar{z}, z), \tag{70a}$$
$$g(r,w) = g(r, w^{-1}). \tag{70b}$$

In addition, the $[\tau \leftrightarrow -\tau]$ symmetry itself implies

$$g(z,\bar{z}) = g(-z, -\bar{z}), \tag{71a}$$
$$g(r,w) = g(r, -w). \tag{71b}$$

C. Generalized free scalar theories

This theory features a fundamental scalar field ϕ of scaling dimension Δ_ϕ, satisfying the equation of motion

$$\Box^{\frac{d}{2}-\Delta_\phi}\phi = 0. \tag{72}$$

Physically, this field follows Gaussian statistics, meaning all higher-point correlation functions factorize into products of two-point functions via Wick's theorem. The OPE expansion can be schematically represented as follows:

$$\phi \times \phi = \mathbb{1} + [\phi\phi]_{n,\,J}. \tag{73}$$

Due to the linearity of the non-interacting theory, the superposition principle holds. Thus, the thermal two-point function on the cylinder $S^1_\beta \times \mathbb{R}^{d-1}$ can be constructed by periodizing the Euclidean space propagator $\langle\phi(x)\phi(0)\rangle_{\mathbb{R}^d} \sim |x|^{-2\Delta_\phi}$ along the thermal circle, i. e., by the method of images,

$$g_{GFF}(\tau,\,x) = \sum_{m=-\infty}^{\infty} \frac{1}{[(\tau+m\beta)^2 + x^2]^{\Delta_\phi}}. \tag{74}$$

At zero spatial separation $x = 0$, (74) can be rewritten using the Hurwitz zeta function $\zeta_H(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$:

$$g_{GFF}(\tau) = \frac{1}{\beta^{2\Delta_\phi}}\left[\zeta_H\left(2\Delta_\phi, \frac{\tau}{\beta}\right) + \zeta_H\left(2\Delta_\phi, 1-\frac{\tau}{\beta}\right)\right]. \tag{75}$$

IV. BOOTSTRAPPING THERMAL CFTS

The goal of this chapter is to determine $\{a_{\mathcal{O}}\}$ from the zero temperature conformal data $\{\Delta_{\mathcal{O}}, J_{\mathcal{O}}, f_{\mathcal{O}\mathcal{O}'\mathcal{O}''}\}$. To do this, we extend the correlator to a meromorphic function in the complex plane, and using the constraints given in section III B, formulate a dispersion relation that the two-point functions must satisfy. Specifically, at zero spatial separation, this dispersion relation allows us to write the form of the two-point function exactly.

A. Analytic bootstrap at zero spatial separation

Let us begin by considering the limit $x \to 0$. This limit makes the problem into a simple one-dimensional problem and allows the OPE to converge in the entire τ range.

1. Dispersion relation

Consider $g(\tau) \propto \frac{1}{\tau^{2\Delta_\phi}}$ as a meromorphic function on the complex plane $\tau \in \mathbb{C}$. Since $g(\tau)$ has a pole at $\tau = 0$, by the KMS condition (69a), $g(\tau)$ has countably many poles at

$$\tau = n\beta, \quad n \in \mathbb{Z}. \tag{76}$$

Δ_ϕ may not be an integer, so branch cuts can be present. Because of the KMS condition (69a) and the time reversal symmetry $[\tau \leftrightarrow -\tau]$, the branch cuts cannot lie in any direction, and can only lie in the same direction as the imaginary axis:

$$\tau = n\beta + it, \quad n \in \mathbb{Z},\, t \in (-\infty,\, 0). \tag{77}$$

Using the analytic structure of the two-point function and Cauchy's integral formula, the two-point function can be rewritten as an exact formula. By Cauchy's integral formula,

$$g(\tau) = \frac{1}{2\pi i}\oint_C d\tau' \frac{g(\tau')}{\tau' - \tau}. \tag{78}$$

Now, let us deform the contour C as shown in Figure 1 and separate the contour integral into the part on the arcs of radius $R \to \infty$ and the remaining part. Then, the two-point function becomes

$$g(\tau) = g_{\rm dr}(\tau) + g_{\rm arcs}(\tau), \tag{79}$$

where

$$g_{\rm arcs}(\tau) = \oint_{C_{\rm arc}} \frac{d\tau'}{2\pi i}\frac{g(\tau')}{\tau' - \tau} \tag{80}$$

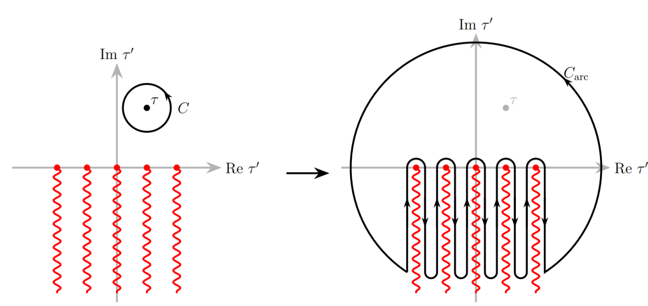

FIG. 1. The contours considered to compute the correlation function in τ via a dispersion relation. The red dots indicate the poles, while the red wavy lines indicate the branch cuts.

and

$$
\begin{aligned}
g_{\rm dr}(\tau) &= \frac{1}{2\pi i}\oint_{C-C_{\rm arc}} d\tau' \frac{g(\tau')}{\tau'-\tau} \\
&= \sum_{m=-\infty}^{\infty}\left(\int_{C_L^m}\frac{d\tau'}{2\pi i}\frac{g(\tau'-\epsilon)}{\tau'-\tau}+\int_{C_R^m}\frac{d\tau'}{2\pi i}\frac{g(\tau'+\epsilon)}{\tau'-\tau}\right) \\
&= \sum_{m=-\infty}^{\infty}\int_{-i\infty}^{0}\frac{d\tau'}{2\pi i}\frac{g(\tau'-\epsilon)-g(\tau'+\epsilon)}{\tau'+m-\tau} \\
&\overset{\text{notation}}{=\!=} \sum_{m=-\infty}^{\infty}\int_{-i\infty}^{0}\frac{d\tau'}{2\pi i}\frac{\mathrm{Disc}\, g(\tau')}{\tau'+m-\tau}
\end{aligned}
\tag{81}
$$

Here, C_L^m and C_R^m denote the left and right parts of the contour encircling the m-th pole, respectively, and $\epsilon \to 0+$. The KMS condition is built into the dispersion relation (81) since the sum runs over all the poles.

2. Discontinuity of thermal conformal blocks

We will plug the OPE (67) into the dispersion relation (81). This allows $g_{\rm dr}(\tau)$ to be written in terms of the discontinuity of the thermal conformal blocks. Therefore, by calculating the discontinuity of each thermal block, we can write $g_{\rm dr}(\tau)$ in an analytic expression. The discontinuity of a thermal block is given by:

$$
\begin{aligned}
&\mathrm{Disc}\tau^{\Delta-2\Delta_\phi} \\
&\quad= |\tau|^{\Delta-2\Delta_\phi}\left(e^{\frac{3}{2}\pi i(\Delta-2\Delta_\phi)}-e^{-\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}\right) \\
&\quad= |\tau|^{\Delta-2\Delta_\phi}e^{\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}2i\sin((\Delta-2\Delta_\phi)\pi).
\end{aligned}
\tag{82}
$$

Therefore, let $\tau' = -iu$, the problem reduces to calculating

$$
I(m,\tau) = e^{\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}\int_0^\infty \frac{du}{2\pi}\frac{u^{\Delta-2\Delta_\phi}}{-iu+m-\tau}. \tag{83}
$$

Through a tedious calculation, we have

$$
\begin{aligned}
I(m,\tau) &= e^{\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}\int_0^\infty\frac{du}{2\pi}\left[\frac{(m-\tau)u^{\Delta-2\Delta_\phi}}{(m-\tau)^2+u^2}+\frac{iu^{\Delta-2\Delta_\phi+1}}{(m-\tau)^2+u^2}\right] \\
&= e^{\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}\left[(m-\tau)\frac{(m-\tau)^{\Delta-2\Delta_\phi-1}}{4\sin\left(\frac{\Delta-2\Delta_\phi+1}{2}\pi\right)}+i\frac{\pi(m-\tau)^{\Delta-2\Delta_\phi}}{4\sin\left(\frac{\Delta-2\Delta_\phi+2}{2}\pi\right)}\right] \\
&= e^{\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}\frac{(m-\tau)^{\Delta-2\Delta_\phi}}{4}\left[\frac{1}{\cos\left(\frac{\Delta-2\Delta_\phi}{2}\pi\right)}-i\frac{1}{\sin\left(\frac{\Delta-2\Delta_\phi}{2}\pi\right)}\right] \\
&= e^{\frac{1}{2}\pi i(\Delta-2\Delta_\phi)}\frac{(m-\tau)^{\Delta-2\Delta_\phi}}{4}\frac{-2ie^{-\frac{1}{2}\pi i(\Delta-\Delta_\phi)}}{\sin((\Delta-2\Delta_\phi)\pi)} \\
&= -i\frac{(m-\tau)^{\Delta-2\Delta_\phi}}{2\sin((\Delta-2\Delta_\phi)\pi)}.
\end{aligned}
\tag{84}
$$

Here, we use the integral formula

$$
\int_0^\infty \frac{x^m}{b^2+x^2}dx = \frac{\pi b^{m-1}}{2\sin\left(\frac{m+1}{2}\pi\right)}.
$$

Using (84), we finally obtain

$$
\begin{aligned}
g_{\rm dr}(\tau) &= \sum_\Delta a_\Delta \sum_{m=-\infty}^{\infty} 2i\sin((\Delta-2\Delta_\phi)\pi)I(m,\tau) \\
&= \sum_\Delta a_\Delta \sum_{m=-\infty}^{\infty}(m-\tau)^{\Delta-2\Delta_\phi} \\
&= \sum_\Delta a_\Delta\left[\zeta_H(2\Delta_\phi-\Delta,\tau)+\zeta_H(2\Delta_\phi-\Delta,1-\tau)\right]
\end{aligned}
\tag{85}
$$

3. Arc contributions

We now estimate the arc parts contained in (79). As a result of the estimation, we prove that the arc contribution $g_{\rm arcs}(\tau)$ is a constant. The reason this estimation is possible is that the analytic structure of the two-point function is already fully reflected in $g_{\rm dr}(\tau)$, so $g_{\rm arcs}(\tau)$ is necessarily an entire function.

To find the bound of $g_{\rm dr}(\tau)$, we will apply the arguments in [9]. Since the thermal geometry $S^1_\beta \times \mathbb{R}^{d-1}$ has a Euclidean signature, we can arbitrarily choose the time direction; in particular, let us perform quantization by choosing one direction in $\mathbb{R}^{d-1}$ as time and identifying the spatial slices as $S^1_\beta \times \mathbb{R}^{d-2}$. Then, the S^1_β compactification can be interpreted as a Kaluza–Klein compactification. In this quantization, the two-point function is given by

$$g(\tau) = \langle\phi(\tau)\phi(0)\rangle_\beta = \langle\Psi|e^{i\tau P_{KK}}|\Psi\rangle, \tag{86}$$

where $|\Psi\rangle = \phi(0)|0\rangle_{S^1_\beta\times\mathbb{R}^{d-2}}$. Let us write the time evolution operator $e^{i\tau P_{KK}}$ as:

$$e^{i\tau P_{KK}} = V^{\frac{1}{2}}UV^{\frac{1}{2}}, \tag{87a}$$
$$V = e^{-\operatorname{Im}(\tau)P_{KK}}, \tag{87b}$$
$$U = e^{-\operatorname{Re}(\tau)P_{KK}}. \tag{87c}$$

Now, applying the Cauchy–Schwarz inequality, we obtain

$$|g(\tau)|^2 \leq \langle\Psi|V^{\frac{1}{2}}V^{\frac{1}{2}}|\Psi\rangle\langle\Psi|V^{\frac{1}{2}}U^\dagger UV^{\frac{1}{2}}|\Psi\rangle = \langle\Psi|V|\Psi\rangle^2. \tag{88}$$

Therefore, we can bound $g(\tau)$ by the real time correlator:

$$|g(\tau)| \leq |g(i\operatorname{Im}\tau)| \tag{89}$$

Moreover, since the real time correlator does not have pole and it is bounded as $|t|\to\infty$ by clustering decomposition or unitarity, $g(\tau)$ is eventually bounded in the imaginary direction. By (85), $g_{\mathrm{dr}}(\tau)$ is polynomially bounded, so $g_{\mathrm{arcs}}(\tau)$ is also polynomially bounded in the imaginary direction. Furthermore, since periodicity also bounds $g_{\mathrm{arcs}}(\tau)$ in the real direction, $g_{\mathrm{arcs}}(\tau)$ is a polynomially bounded entire function. Cauchy's estimate gives

$$g^{(n)}_{\mathrm{arcs}}(0) \leq \frac{n!k|z|^m}{R^n}, \quad \forall z\in\{z\in\mathbb{C}\,|\,|z|=R\} \tag{90}$$

for some $k\in\mathbb{R}_{>0}$ and $m\in\mathbb{Z}_{\geq 0}$. For $n>m$, letting $R\to\infty$, we have $|g^{(n)}_{\mathrm{arcs}}(0)| = 0$. Consider the Taylor series of $g_{\mathrm{arcs}}(\tau)$ around 0, we clearly see that the $g_{\mathrm{arcs}}(\tau)$ is polynomial. Using periodicity once more, we conclude

$$g_{\mathrm{arcs}}(\tau) = \kappa, \tag{91}$$

for some constant $\kappa\in\mathbb{R}$.

4. A formula for $g(\tau)$

Using the results of Section 2 and Section 3, we rewrite $g(\tau)$ as

$$g(\tau) = \sum_\Delta a_\Delta\left[\zeta_H(2\Delta_\phi-\Delta,\tau)+\zeta_H(2\Delta_\phi-\Delta,1-\tau)\right]+\kappa, \tag{92a}$$

$$g(\tau) = \sum_\Delta \frac{a_\Delta}{\beta^\Delta}\left[\zeta_H(2\Delta_\phi-\Delta,\frac{\tau}{\beta})+\zeta_H(2\Delta_\phi-\Delta,1-\frac{\tau}{\beta})\right]+\kappa, \tag{92b}$$

for (92b), we reintroduce β through dimensional analysis. Here, one can easily verify that $g(\tau)$ becomes the zero-temperature two-point function in the limit $\beta\to\infty$. This formula inherently encodes the KMS condition. Furthermore, considering (75), we conclude that any thermal two-point function at zero spatial separation is an expansion in terms of GFF correlators $g_{GFF}(\tau)$, where the coefficients are the reduced OPE coefficients.

The GFF correlators $g_{GFF}(\tau)$ vanish when $\Delta-2\Delta_\phi\in 2\mathbb{Z}_{>0}$. That is, double-twist operators satisfying $\Delta_{n,J}-2\Delta_\phi\in 2\mathbb{Z}_{>0}$ cannot contribute to this expansion. Therefore, for a theory where the heavy spectrum consists solely of double-twist operators, we can write the thermal two-point function in terms of light operators.

B. Analytic bootstrap at non-zero spatial separation

We now extend the analysis to non-zero spatial separation ($x\neq 0$). Similar to the zero separation case, we will use the strategy of reconstructing the correlator by plugging the t-channel OPE into the dispersion relation. Since the dispersion relation for non-zero separation was established in [9], we first provide a brief review of these results.

1. Dispersion relation

Analogous to the derivation of the dispersion relation in the complex τ-plane in Section IV A 1, we can derive the dispersion relation for $g(r,w)$ by fixing r, determining its analytic structure in the complex w-plane, and applying contour deformation.

For OPE convergent, we first assume that $0<r<$

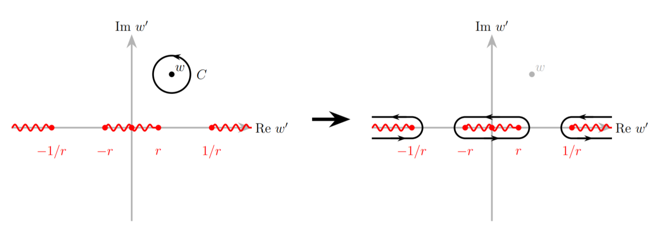

FIG. 2. The contours considered to compute the correlation function in w via a dispersion relation.

1. Originally, since $w = z/\bar{z}$, w should belong to the unit circle $\{w \in \mathbb{C} \,|\, |w| = 1\}$, but we consider w as a complex variable defined over the entire complex w-plane. Then, the two-point function $g(r, w)$ has poles at $w = 0, \pm r, \pm 1/r$ because of the KMS condition (69b). By (70b) and (71b), the brunch cuts has to be the intervals $(-\infty, -1/r)$, $(-r, r)$, $(1/r, \infty)$ as Figure 2.

As same as the $x = 0$ case, by Cauchy's integral formula,

$$g(\tau) = \frac{1}{2\pi i} \oint_C d\tau' \frac{g(\tau')}{\tau' - \tau}. \tag{93}$$

By deforming the contour as shown in Figure 2 and retaining only the contribution from the discontinuity across the branch cuts, one can obtain:

$$g_{\rm dr}(r, w) = \int_0^r \frac{dw'}{2\pi i} \frac{w^2(1 - w'^4)\,{\rm Disc}\, g(r, w')}{w'(w' - w)(w' + w)(1 - w^2 w'^2)}, \tag{94}$$

where

$${\rm Disc}\, g(r, w) := \frac{1}{i}(g(r, w + i\epsilon) - g(r, w - i\epsilon)). \tag{95}$$

For a detailed derivation of this dispersion relation (94) and a discussion on its equivalence to the Lorentzian inversion formula introduced in [9], please refer to Section 5 of [14].

Following (79), let us collect the remaining parts of the two-point function, specifically the contributions from the arcs and the residue at $w = 0$, into a single function $g_{\rm arcs}$:

$$g(\tau) = g_{\rm dr}(\tau) + g_{\rm arcs}(\tau), \tag{96}$$

We define it in this manner because the residue contribution at $w = 0$ corresponds to the identity operator contribution, which is not our primary focus here.

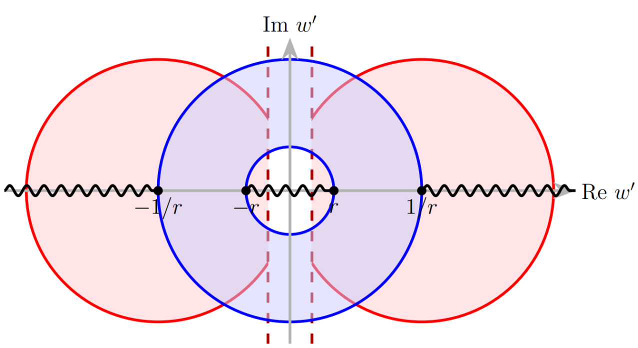

FIG. 3. The s-channel OPE (expansion around $z = 0$, $\bar{z} = 0$) converges within the annulus shaded in blue ($|z| < 1$ and $|\bar{z}| < 1$), excluding the intervals $[-\infty, -1/r] \cup [-r, r] \cup [1/r, \infty]$, while the t-channel OPE (expansion around $z = 1$, $\bar{z} = 1$) converges within the two disks shaded in red ($|1 - z| < 1$ and $|1 - \bar{z}| < 1$), with the same intervals removed.

2. *OPE and out-of-OPE contributions*

A major difference between the non-zero and zero spatial separation cases is that the OPE does not converge on the entire w-plane, but only within a specific region. As shown in Figure 3, the OPE converges only in a finite region of the w-plane. The critical problem is that our contour integral passes through a region where the OPE does not converge.

To obtain the OPE contribution to $g_{\rm dr}$, we substitute the t-channel OPE into the dispersion relation (94) and compute the discontinuity of each thermal conformal block. Each thermal block schematically takes the form

$$f_{\Delta,J}(1-z, 1-\bar{z}) \sim \sum_{\Delta < 2\Delta_\phi} \frac{1}{(1 - \bar{z})^{\Delta_\phi - \Delta/2}} + \text{regular}, \tag{97}$$

and the discontinuity of each term in the series is given by

$${\rm Disc}(1 - \bar{z})^\alpha = \begin{cases} 0 & \text{if } \alpha \in \mathbb{Z}_{>0}, \\ 2i \sin(\alpha\pi)\Theta({\rm Re}\,\bar{z} - 1) & \text{if } \alpha \notin \mathbb{Z}_{>0}, \end{cases} \tag{98}$$

analogous to the previous calculation (82). This means that, as the zero spatial separation case, the correlator can be reconstructed from a finite number of light operators.

As discussed previously, the OPE does not converge over the entire integration contour. Indeed, this phenomenon is manifested in the fact that the dispersion relation (94), unlike (81), does not a priori satisfy the KMS condition. As already observed in [9], we must distinguish between the following two types of contributions:

- *OPE contributions (singular region).* These contributions arise from the region near the branch point at $\bar{z} = 1$, where the discontinuity exhibits a power-law singularity. The Lorentzian inversion integral takes the form:

$$I_{\rm OPE}(J) \sim \int_1^\infty \frac{d\bar{z}}{\bar{z}} \bar{z}^{\Delta_\phi - \bar{h} - m} {\rm Disc}[(1 - \bar{z})^c], \tag{99}$$

where $\bar{h} = (\Delta + J)/2$ and $m \in \mathbb{Z}_{\geq 0}$. In the large spin limit ($J \to \infty$), with $\bar{h} \approx J$, the kernel behaves as $\bar{z}^{-J}$. We apply Watson's lemma by substituting $t = \bar{z} - 1$:

$$\begin{aligned} I_{\rm OPE}(J) &\sim \int_0^\infty dt\,(1 + t)^{-J}\, t^c \approx \int_0^\infty dt\, e^{-Jt}\, t^c \\ &= J^{-(c+1)} \int_0^\infty du\, e^{-u} u^c \quad (\text{where } u = Jt) \\ &\sim \Gamma(c + 1) J^{-(c+1)} \end{aligned} \tag{100}$$

This confirms that contributions from the OPE region are polynomially suppressed in spin, J^{-c-1}.

- *Out-of-OPE contributions (regular region).* These contributions come from the integration range away from the singularity (e.g., $\bar{z} \geq z_0 > 1$, typically $z_0 = 2$), where the function $f(\bar{z})$ is regular. The integral is given by:

$$I_{\text{out}}(J) \sim \int_{z_0}^{\infty} \frac{d\bar{z}}{\bar{z}} \bar{z}^{\Delta_\phi - \bar{h} - m} f(\bar{z}) \sim \int_{z_0}^{\infty} d\bar{z}\, \bar{z}^{-J} f(\bar{z}) \tag{101}$$

Since the integrand is analytic and decaying, we apply Laplace's method. The integral is dominated by the lower bound z_0:

$$\begin{aligned} I_{\text{out}}(J) &= \int_{z_0}^{\infty} d\bar{z}\, e^{-J \ln \bar{z}} f(\bar{z}) \\ &\approx f(z_0) \int_{z_0}^{\infty} d\bar{z}\, \bar{z}^{-J} = f(z_0) \left[\frac{\bar{z}^{-J+1}}{-J+1} \right]_{z_0}^{\infty} \\ &\sim \frac{f(z_0)}{J} z_0^{-J} \end{aligned} \tag{102}$$

For $z_0 = 2$, this leads to an exponential suppression 2^{-J}. These non-perturbative corrections are invisible to the standard perturbative $1/J$ expansion, i.e., they are not captured by the dispersion relation (94).

Based on this distinction, we decompose the thermal two-point function into two parts:

$$g(z,\bar{z}) = g_{\text{OPE}}(z,\bar{z}) + g_{\text{out-of-OPE}}(z,\bar{z}) \tag{103}$$

where g_{OPE} corresponds to the dispersion relation and the arc completion, while $g_{\text{out-of-OPE}}$ represents the non-perturbative terms that are not captured by them.

3. *Consistency conditions*

Our goal is to reconstruct $g_{\text{out-of-OPE}}(z,\bar{z})$. To this end, we first introduce the consistency conditions that $g_{\text{out-of-OPE}}(z,\bar{z})$ (or $g(z,\bar{z})$) must satisfy, and then reconstruct $g_{\text{out-of-OPE}}(z,\bar{z})$ by imposing these conditions. These conditions rely on the uniqueness theorem proposed in [14, 24]. Although this theorem is incomplete, our previous analysis has already established the discontinuity of the two-point function across the branch cut. Therefore, in principle, these consistency conditions should allow us to reconstruct the two-point function almost perfectly.

1. *KMS condition.* The most fundamental property that a thermal correlator must satisfy is the KMS condition 69a.

2. *Analytic structure.* $g(z,\bar{z})$ must have the analytic structure depicted in Figure 2.

3. *Consistency with the OPE.* The thermal two-point function must be consistent with the operator product expansion. Specifically, with the exception of the low-spin operators contributing to g_{arcs}, the dispersion relation part must also agree with the OPE.

4. *Regge boundedness.* The thermal two-point function must be polynomially bounded in the Regge limit

$$w \to \infty, \quad r \text{ fixed.} \tag{104}$$

5. *Clustering decomposition at large distances.* In the limit $x \to \infty$, the two-point function must satisfy the cluster decomposition property. Generally, for theories without extensive symmetry structures, the following exponential decay is expected:

$$g(z,\bar{z}) \overset{x\to\infty}{\longrightarrow} \langle\phi\rangle_\beta^2 + O(e^{-m_{\text{th}} x}), \tag{105}$$

where $m_{\text{th}} \propto 1/\beta$ is the thermal mass of the theory.

4. *The generalized method of images*

Motivated by the complex τ-plane at zero spatial separation, our strategy is to explicitly enforce KMS invariance on the two-point function by hand. For this purpose, we rely on the observation that an arbitrary function $f(\tau)$ can generate a periodic function $\widetilde{f}(\tau)$:

$$\widetilde{f}(\tau) = \sum_{m\in\mathbb{Z}} f(\tau - m). \tag{106}$$

The reverse is also true: any periodic function $\widetilde{f}$ admits the same decomposition. Specifically, suppose the Fourier series of $\widetilde{f}(\tau)$ is given by

$$\widetilde{f}(\tau) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k \tau}. \tag{107}$$

Then, by identifying an $f(\tau)$ such that $\hat{f}(k) = c_k$ for all $k \in \mathbb{Z}$, we can write $\widetilde{f}(\tau)$ in the form of (106). Although such $f(\tau)$ is not unique, this does not affect our argument. Based on this observation, we introduce the following Ansatz:

$$g(z,\bar{z}) = \frac{1}{2}\sum_{m\in\mathbb{Z}} g_{\text{dr}}(z-m,\bar{z}-m) + g_{\text{arcs}}(z,\bar{z}), \tag{108}$$

where the factor $1/2$ is introduced to compensate for the double counting of the $\pm m$ terms, as the dispersion relation already reflects the $w \leftrightarrow -w$ symmetry.

This Ansatz is referred to as the *generalized method of images* by the authors of [1]. It manifestly satisfies the consistency conditions listed in Section IV B 3. In particular, the KMS condition and the analytic structure follow immediately from the fact that (108) is merely a summation of the dispersion relation constructed to satisfy the KMS condition. Furthermore, this Ansatz is consistent with the OPE. This is because, for precisely the same reason as in the calculation of (102), the terms with $m \neq 0$ correspond solely to exponential corrections related to n^{-J}. This can also be interpreted as a consequence of the shift $z \mapsto z - m$, which extends the region of convergence of the OPE.

The demonstration of Regge boundedness for the Ansatz rests upon the analytic properties of $g_{dr}(z, \bar{z})$. The central premise is that this function is polynomially bounded in the Regge limit. This boundedness is preserved under the thermal shifts inherent to the image method. Since every image term $g_{dr}(z - m, \bar{z} - m)$ is generated by evaluating the same integral representation at shifted arguments, each term inherits the asymptotic stability of the original $m = 0$ term. Consequently, the full correlator maintains the required asymptotic behavior, because the contribution of the image term to the thermal OPE coefficients scales as n^{-J}.

However, the cluster decomposition property is not automatically guaranteed by this Ansatz. Therefore, we must verify the clustering condition on a case-by-case basis; specifically, g_{arcs} must be reconstructed to satisfy this condition.

C. Applications

1. Generalized free theories

From the discussions in Sections III C and IV A 4, it is evident that our method can reconstruct the two-point function of a generalized free scalar theory at zero spatial separation. We now proceed to test the validity of the generalized method of images for the case of non-zero spatial separation.

Since the identity operator provides the sole contribution to the OPE, we need only calculate g_{dr} for the identity operator. Using its contribution to the discontinuity,

$$\begin{aligned}\text{Disc}[\mathbb{1}] = &-2i\sin(\pi\Delta_\phi)r^{2\Delta_\phi}\left(\frac{r}{w'}-1\right)^{-\Delta_\phi} \\ &\times (1-rw')^{-\Delta_\phi}\,\Theta\left(\text{Re}\frac{r}{w'}-1\right),\end{aligned} \tag{109}$$

we calculate $g_{\text{dr}}(z, \bar{z})$ to be

$$g_{\text{dr}}(z,\bar{z}) = \frac{1}{(z-1)^{\Delta_\phi}(\bar{z}-1)^{\Delta_\phi}} + \frac{1}{(z+1)^{\Delta_\phi}(\bar{z}+1)^{\Delta_\phi}} \tag{110}$$

One can clearly see that this expression does not satisfy the KMS condition.

Applying the generalized method of images to this expression yields the following standard result:

$$g(z,\bar{z}) = \sum_{m\in\mathbb{Z}} \frac{1}{(z-m)^{\Delta_\phi}(\bar{z}-m)^{\Delta_\phi}}. \tag{111}$$

This demonstrates that the generalized method of images works well in the simplest GFF case.

2. O(N) model

In this section, we consider the O(N) model. Our analysis centers on the ε-expansion, performed perturbatively around $d = 4$.

The $O(N)$ model is defined by the standard Ginzburg-Landau action with quartic interaction:

$$\mathcal{A} = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial_\mu\phi_i)^2 + \frac{\lambda}{4!}(\phi_i\phi_i)^2\right], \tag{112}$$

with the index i running from 1 to N, and the upper critical dimension set at $d = 4$. In $d = 4 - \varepsilon$, the theory admits a non-trivial Wilson–Fisher fixed point with a critical coupling $\lambda_\star \sim \varepsilon 1$, rendering the model accessible via perturbative expansion [25].

Here, we apply bootstrap methods to determine the thermal two-point function $\langle\phi\phi\rangle_\beta$ up to the first order in ε. At the first order, the operator does not acquire an anomalous dimension, and its scaling behavior matches that of the free theory:

$$\Delta_\phi = 1 - \frac{\varepsilon}{2} + O(\varepsilon^2). \tag{113}$$

A scaling analysis indicates that at leading order, only the identity $\mathbb{1}$, the higher-spin currents $[\phi\phi]_{0,J}$, and $[\phi\phi]_{1,J}$ are relevant. Higher order operators ($n > 1$) are suppressed by powers of ε due to the equations of motion. Since the OPE coefficients for $[\phi\phi]_{1,J}$ are of order $O(\varepsilon)$, we effectively only need to track the corrections to the $[\phi\phi]_{0,J}$ sector.

Since the OPE coefficients for $[\phi\phi]_{1,J}$ start at order $O(\epsilon)$, we essentially only need to consider corrections to the $[\phi\phi]_{0,J}$ sector. Within this family, only the ϕ^2 operator receives an anomalous dimension at this order, meaning that the majority of terms in the OPE simply vanish.

a. Zero spatial separation. Based on the previous discussion, the two-point function in the complex τ-plane is consists of only two contributions:

$$g(\tau) = g^{(\mathbb{1})}(\tau) + g^{(\phi^2)}(\tau). \tag{114}$$

The contribution from the identity operator is

$$g^{(\mathbb{1})}(\tau) = \zeta_H(2-\varepsilon,\tau) + \zeta_H(2-\varepsilon, 1-\tau). \tag{115}$$

The ϕ^2 contribution is given by

$$\begin{aligned} g^{(\phi^2)}(\tau) &= \zeta_H\left(-\gamma_{\phi^2},\tau\right) + \zeta_H\left(-\gamma_{\phi^2}, 1-\tau\right) \\ &= -a_2^{(0)}\gamma_{\phi^2}\log\left[\frac{\csc(\pi\tau)}{2}\right] + O(\varepsilon^2),\end{aligned} \tag{116}$$

Here, $a_2^{(0)}$ represents the coefficient $a_{\Delta=2}$ in the $4d$ free theory and γ_{ϕ^2} denotes the anomalous dimension of the ϕ^2 operator at the first order in ε, which is [26]

$$\gamma_{\phi^2} = \varepsilon\frac{N+2}{N+8}. \tag{117}$$

b. Non-zero spatial separation. The contribution of the identity to $g_{\rm dr}$ is (110). The first-order terms in ε is

$$g_{\rm dr}^{(\mathbb{1})} = \frac{(z\bar{z}+1)}{(z^2-1)(\bar{z}^2-1)}\log\left(\frac{(z-1)(\bar{z}-1)}{z\bar{z}}\right) + \frac{1}{(z+1)(\bar{z}+1)}\left(\tanh^{-1}(z)+\tanh^{-1}(\bar{z})\right). \tag{118}$$

The discontinuity for the ϕ^2 contribution is

$$\mathrm{Disc}[\phi^2] = i\pi\,\gamma_\phi\, z\bar{z}\,\Theta(\mathrm{Re}(\bar{z})-1). \tag{119}$$

Therefore, through a calculation analogous to (110), we obtain

$$g_{\rm dr}^{(\phi^2)} = \frac{1}{2}a_{\phi^2}^{(0)}\gamma_{\phi^2}\left[\log(1-z^2)+\log(1-\bar{z}^2)\right], \tag{120}$$

where $a_{\phi^2}^{(0)}$ denotes the OPE coefficient at order $O(1)$.

Furthermore, solving the KMS condition (69a) implies

$$a_{\phi^2}^{(0)} = \frac{2\varepsilon}{\varepsilon-\gamma_{\phi^2}}. \tag{121}$$

However, this contradicts the fact that the free theory coefficient $a_{\phi^2}^{(0)}$ should not depend on γ_{ϕ^2}. We interpret this as evidence that ϕ^2 contributes to the arcs, and thus cannot be fully constrained by the dispersion relation alone.

c. The generalized method of images. Our interest lies specifically in the difference $g(\tau)-g_{\rm free}(\tau)$, as the free part is exactly solvable in $4-\varepsilon$ dimensions. The generalized method of images yields:

$$\frac{1}{2}\sum_{m=-\infty}^{\infty} g_{\rm dr}^{(\phi^2)}(z-m,\bar{z}-m) = \frac{a_{\phi^2}^{(0)}}{4}\gamma_{\phi^2}\sum_{m=-\infty}^{\infty}\left[\log(1-(z-m)^2)+\log(1-(\bar{z}-m)^2)\right], \tag{122}$$

This series diverges on the real axis. Nevertheless, it can be evaluated via analytic continuation. Using

$$\log(1+(x-m)) = -\frac{\partial}{\partial s}\frac{1}{(1+x-m)^s}\bigg|_{s\to 0}, \tag{123}$$

we find that the sum can be regularized using the ζ_H-function:

$$\sum_{m=-\infty}^{\infty}\log(1+(x-m)) = -\log\left[\frac{\csc(\pi x)}{2}\right]. \tag{124}$$

Applying this result, we conclude

$$\begin{aligned} g(z,\bar{z})-g_{\rm free}(z,\bar{z}) &= \frac{1}{2}\sum_{m=-\infty}^{\infty} g_{\rm dr}^{(\phi^2)}(z-m,\bar{z}-m) \\ &= -\frac{1}{2}a_{\phi^2}^{(0)}\gamma_{\phi^2}\log\left[\frac{\csc(\pi z)\csc(\pi\bar{z})}{4}\right]. \end{aligned} \tag{125}$$

D. A thermal semi-analytic bootstrap

In this work, we improve upon the numerical bootstrap method proposed in [27] and reproduce the numerical results reported in [2]. In [2, 27], the authors utilized Tauberian theorems to argue for the following asymptotic behavior:

$$a_{\widetilde{\Delta}} \overset{\widetilde{\Delta}\gg 1}{\approx} \frac{2\widetilde{\Delta}^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\left(1+\frac{c_1}{\widetilde{\Delta}}+O\left(\frac{1}{\widetilde{\Delta}^2}\right)\right), \tag{126}$$

with a unknown coefficient c_1. While this approximation yields satisfactory results, in this thesis, we derive a more improved form of the asymptotic behavior. Moreover, since the derivation in [27] was not mathematically perfect, this thesis serves to provide a rigorous proof of (126).

1. Asymptotic behavior of heavy operators

Here, we consider only the case of zero spatial separation. Starting from (92a), we expand the Hurwitz ζ-function around $\tau=0$ as

$$\begin{aligned} &\zeta_H(s,\tau)+\zeta_H(s,1-\tau) \\ &\quad = \frac{1}{\tau^s}+\sum_{k=0}^{\infty}\left[1+(-1)^k\right]\binom{s+k-1}{k}\zeta(k+s)\tau^k, \end{aligned} \tag{127}$$

where $s=2\Delta_\phi-\Delta$. Substituting this back into (92a), we have

$$g(\tau) = \sum_{\Delta} a_\Delta \tau^{-s} + \sum_{k=0}^{\infty}\left[\sum_{\Delta} a_\Delta\left[1+(-1)^k\right]\binom{s+k-1}{k}\zeta(k+s)\right]\tau^k + \kappa \tag{128}$$

This expression (128) must be essentially equivalent to the OPE (67). From channel duality, since the heavy operator contribution in the s-channel must match the light operator contribution in the t-channel, the following approximation holds:

$$\sum_{\widetilde{\Delta}\gg 1} a_{\widetilde{\Delta}}\tau^{\widetilde{\Delta}-2\Delta_\phi} \cong \sum_{k=0}^{\infty}\left[\sum_{\Delta \text{ light}} a_\Delta[\cdots]\tau^k\right] \tag{129}$$

By comparing the coefficients on both sides of (129), we easily find

$$a_{\widetilde{\Delta}} = \sum_{\Delta \text{ light}} a_\Delta \left[1+(-1)^{\widetilde{\Delta}-2\Delta_\phi}\right]\binom{\widetilde{\Delta}-\Delta-1}{\widetilde{\Delta}-2\Delta_\phi}\zeta(\widetilde{\Delta}-\Delta) \tag{130}$$

In particular, in the limit $\widetilde{\Delta}\to\infty$, the following three asymptotic behaviors hold:

$$\binom{\widetilde{\Delta}-\Delta-1}{\widetilde{\Delta}-2\Delta_\phi} \approx \frac{\widetilde{\Delta}^{2\Delta_\phi-\Delta-1}}{\Gamma(2\Delta_\phi-\Delta)}, \tag{131}$$

$$\zeta(\widetilde{\Delta}-\Delta)\to 1, \tag{132}$$

$$\log\frac{\Gamma(\widetilde{\Delta}-\Delta)}{\Gamma(\widetilde{\Delta}-2\Delta_\phi+1)} \approx (2\Delta_\phi-\Delta-1)\log\widetilde{\Delta} + \sum_{k=1}^{n}\frac{b_k^\Delta}{\widetilde{\Delta}^k} + O(\widetilde{\Delta}^{-n-1}), \tag{133}$$

where

$$b_k^\Delta = \frac{B_{k+1}(1-2\Delta_\phi)-B_{k+1}(\Delta)}{k(k+1)}, \tag{134}$$

with the Bernoulli polynomial B_k. Consequently, we obtain that the contribution of the operator of dimension Δ to $a_{\widetilde{\Delta}}$:

$$a_{\widetilde{\Delta}}^{(\Delta)} \overset{\widetilde{\Delta}\gg 1}{\sim} a_\Delta \frac{2\widetilde{\Delta}^{2\Delta_\phi-\Delta-1}}{\Gamma(2\Delta_\phi-\Delta)}\left(1+\frac{c_1^\Delta}{\widetilde{\Delta}}+\frac{c_2^\Delta}{\widetilde{\Delta}^2}+\cdots\right), \tag{135}$$

where $c_m^\Delta = \frac{1}{m!}B_m(b_1^\Delta, 1!b_2^\Delta, \cdots, (m-1)!b_m^\Delta)$ is a complete Bell polynomial. Here, the summation is approximated by a sum over only double-twist operators with even integer dimensions. For example, the first two c_m^Δ are given by

$$c_1^\Delta = \frac{(\Delta+2\Delta_\phi)(\Delta-2\Delta_\phi+1)}{2}, \tag{136a}$$

$$c_2^\Delta = \frac{(\Delta-2\Delta_\phi+2)\left(3\Delta^2+\Delta(12\Delta_\phi+1)+2\Delta_\phi(6\Delta_\phi-1)\right)(\Delta-2\Delta_\phi+1)}{24}. \tag{136b}$$

Eq. (135) corrects the leading Tauberian approximation for (126). This correction is driven by the inversion of operators with dimension Δ. The identity operator makes the primary contribution:

$$a_{\widetilde{\Delta}} \overset{\widetilde{\Delta}\gg 1}{\sim} \frac{2\widetilde{\Delta}^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\left[1-\frac{\Delta_\phi(2\Delta_\phi-1)}{\widetilde{\Delta}} + \frac{\Delta_\phi(\Delta_\phi-1)(2\Delta_\phi-1)(6\Delta_\phi-1)}{6\,\widetilde{\Delta}^2}+\ldots\right] + \ldots. \tag{137}$$

This equation matches the Tauberian asymptotic derived in [27]. Therefore, we conclude that we have proven the Tauberian asymptotic via the analytic bootstrap.

2. O(N) *models for* $N = 1, 2, 3$

Here, we adopt the approach of [2] to numerically compute the OPE coefficients of $O(N)$ models ($N = 1, 2, 3$). We begin by deriving a sum rule, which serves as a stringent constraint on these coefficients. Using the asymptotic behavior of a_Δ from Section IV D 1, we then convert this sum rule into an approximate minimization problem involving a specific cost function. Finally, we solve this minimization problem numerically to find the coefficients.

a. A derivation of the sum rules. To apply the KMS condition easily, we first look at the OPE of $g(\beta/2\pm\tau, x)$ and expand it again in terms of τ. In this case, the KMS condition corresponds to the τ-reversal symmetry: $g(\beta/2+\tau,x) = g(\beta/2-\tau,x)$. For simplicity, let

$$A_{\mathcal{O}}^k = \frac{a_{\mathcal{O}}}{\beta^\Delta}(-1)^k\frac{\Gamma(J-k+\nu)}{\Gamma(\nu)k!(J-2k)!}2^{J-2k}, \tag{138}$$

and let $h = \Delta - J$, then,

$$
\begin{aligned}
g(\beta/2\pm\tau,x) &= \sum_{\mathcal{O}}\sum_{k=0}^{\lfloor J/2\rfloor} A^k_{\mathcal{O}}\left|(\beta/2\pm\tau)^2+x^2\right|^{\frac{h-2\Delta_\phi+2k}{2}}(\beta/2\pm\tau)^{J-2k} \\
&= \sum_{\mathcal{O}}\sum_{k=0}^{\lfloor J/2\rfloor}\sum_{n=0}^{\infty} A^k_{\mathcal{O}}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n}(\beta/2\pm\tau)^{\Delta-J-2\Delta_\phi+2k-2n}x^{2n}(\beta/2\pm\tau)^{J-2k} \\
&= \sum_{\mathcal{O}}\sum_{k=0}^{\lfloor J/2\rfloor}\sum_{n=0}^{\infty} A^k_{\mathcal{O}}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n}(\beta/2\pm\tau)^{\Delta-2\Delta_\phi-2n}x^{2n} \\
&= \sum_{\mathcal{O}}\sum_{k=0}^{\lfloor J/2\rfloor}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A^k_{\mathcal{O}}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n}\binom{\Delta-2\Delta_\phi-2n}{m}\left(\frac{\beta}{2}\right)^{\Delta-2\Delta_\phi-2n-m}(\pm\tau)^m x^{2n}.
\end{aligned}
\tag{139}
$$

By the KMS condition, terms with odd powers of τ in 139 must vanish, namely, for all $m\in 2\mathbb{Z}_{\geq 0}+1$ and x,

$$
\sum_{\mathcal{O}}\sum_{k=0}^{\lfloor J/2\rfloor}\sum_{n=0}^{\infty} A^k_{\mathcal{O}}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n} \times\binom{\Delta-2\Delta_\phi-2n}{m}\left(\frac{\beta}{2}\right)^{\Delta-2\Delta_\phi-2n} x^{2n}=0 \tag{140}
$$

Therefore, for all $n\in\mathbb{Z}_{\geq 0}$ and $m\in 2\mathbb{Z}_{\geq 0}+1$,

$$
\sum_{\mathcal{O}}\sum_{k=0}^{\lfloor J/2\rfloor} A^k_{\mathcal{O}}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n} \times\binom{\Delta-2\Delta_\phi-2n}{m}\left(\frac{\beta}{2}\right)^{\Delta-2\Delta_\phi}=0. \tag{141}
$$

This gives a set of sum rules

$$
\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}} f_{\mathcal{O}\phi\phi} F_{m,n}(h,J)=0, \tag{142}
$$

for all $n\in\mathbb{Z}_{\geq 0}$ and $m\in 2\mathbb{Z}_{\geq 0}+1$, where

$$
\begin{aligned}
F_{m,n}(h,J) =& \frac{1}{2^{h+J}}\binom{\frac{h-2\Delta_\phi}{2}}{n}\binom{h+J-2\Delta_\phi-2n}{m} \\
&\times {}_3F_2\left[\left.\begin{matrix}\frac{1-J}{2},-\frac{J}{2},\frac{h}{2}-\Delta_\phi+1\\ \frac{h}{2}-\Delta_\phi-n+1,-J-\nu+1\end{matrix}\right|1\right],
\end{aligned}
\tag{143}
$$

and we set the normalization constant $c_{\mathcal{O}}=1$. For our purposes, let us simplify these sum rules to zero spatial separation $x=0$. After some calculations, we get a simpler form of sum rules

$$
-\frac{\Gamma(2\Delta_\phi+m)}{\Gamma(2\Delta_\phi)}+\sum_{\Delta}\frac{a_\Delta}{2^\Delta}\frac{\Gamma(\Delta-2\Delta_\phi+1)}{\Gamma(\Delta-2\Delta_\phi-m+1)}=0, \tag{144}
$$

where $m\in 2\mathbb{Z}_{\geq 0}+1$ and the sum runs over all $\Delta\neq 0$, with the first term corresponding to the identity contribution. This sum rule imposes countably many constraints on the thermal conformal data a_Δ. Finally, let us define the kernel F as

$$
\mathsf{F}(\Delta,\Delta_\phi,m)=\frac{1}{2^\Delta}\frac{\Gamma(\Delta-2\Delta_\phi+1)}{\Gamma(\Delta-2\Delta_\phi-m+1)}. \tag{145}
$$

b. O(N) *models.* We split the sum rule into contributions from light and heavy operators:

$$
\mathsf{f}(m)=\sum_{\Delta\leq\Delta_{\max}} a_\Delta\mathsf{F}(\Delta,\Delta_\phi,m)+\sum_{\Delta>\Delta_{\max}} a_\Delta\mathsf{F}(\Delta,\Delta_\phi,m) \tag{146}
$$

Exploiting the asymptotic behavior, we approximate the heavy operator contribution as $a_\Delta\sim\frac{2\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\left(1+\frac{c_1}{\Delta}\right)$ for double-twist operators with dimensions $\Delta_{n,J}=2\Delta_\phi+2n+J$ ($n\in\mathbb{Z}_{\geq 0}$, $J\in 2\mathbb{Z}_{\geq 0}$). This converts the sum rule into a minimization problem for the following cost function:

$$
\eta(\{\omega_i\})=\sum_{m\leq m_{\max}}\omega_m\mathsf{f}(m)^2, \tag{147}
$$

where $m_{\max}$ represents the number of sum rule constraints and $\omega_i\in(0,1)$ are random number wights. Following [2], we set $m_{\max}=7$ and $\Delta_{\max}=4$. Tables II, III, and IV show the OPE coefficients determined by this procedure. We compare our results with those in [2]; for the $N=1$ case, we also compare them with results from the large spin perturbation [10].

TABLE II. Thermal OPE coefficients a_Δ for light operators in the 3d Ising model. Comparison with the previous Tauberian results (PR) and the large spin perturbation (LSP).

Operator	This work	PR [2]	LSP [10]
ϵ	0.672(24)	0.75(15)	0.672(74)
$T_{\mu\nu}$	2.039(54)	2.092(13)	1.96(2)
ϵ'	0.177(22)	0.17(2)	0.17(2)
c_1	−0.0809(23)	−0.065	–

TABLE III. Thermal OPE coefficients a_Δ for light operators in the XY model ($N = 2$).

Operator	This work	PR [2]
$\phi_i \phi_i$	0.644(22)	0.73(14)
$T_{\mu\nu}$	1.991(68)	1.90(8)
$(\phi_i\phi_i)^2$	0.184(25)	0.20(7)
c_1	−0.0690(18)	−0.0539

TABLE IV. Thermal OPE coefficients a_Δ for light operators in the Heisenberg model ($N = 3$).

Operator	This work	PR [2]
$\phi_i \phi_i$	0.678(26)	0.76(14)
$T_{\mu\nu}$	1.91(63)	1.81(8)
$(\phi_i\phi_i)^2$	0.189(20)	0.21(7)
c_1	−0.0611(17)	−0.0471

V. CONCLUSIONS AND FUTURE WORK

In this thesis, we reviewed the analytic thermal bootstrap program [1] and successfully reproduced the results of [2]. We focused on three main achievements regarding thermal two-point functions. First, for the case of zero spatial separation, we demonstrated that the thermal two-point function can be expanded using generalized free field (GFF) correlators. Second, for the case of non-zero spatial separation, we proposed a new approach called the *generalized method of images*. We applied this method to O(N) model and successfully calculated the contributions from the ϕ^2 operator. Third, we mathematically proved the asymptotic behavior of heavy operators. This provides a rigorous analytic proof for the approximation used in previous studies [27]. Using this proven behavior, we numerically determined the OPE coefficients for $O(N)$ models with $N = 1$, 2 and 3, which matched well with the original results of [2].

The thermal bootstrap program is still in its early stages of development, presenting numerous open problems. Among these, we focus on three future directions of particular interest:

- ⋆ *Extension to holographic theories.* A natural and significant extension of this work is to apply the analytic thermal bootstrap to holographic theories [6]. In the context of the AdS/CFT correspondence, the thermal properties of the boundary CFT provide a window into the physics of black holes in the bulk [4, 28]. A particularly compelling target is the 4d $\mathcal{N} = 2$ partition function in the Nekrasov–Shatashvili limit. This case is of special interest because the thermal two-point function in momentum space is known from the gravity side [29], providing a benchmark for direct comparison. However, to date, no study has fully determined this two-point function relying solely on the conformal bootstrap [6, 7]. Applying our framework to this limit would therefore fill this gap and allow for a rigorous test of the thermal bootstrap against holographic predictions.
- ⋆ *Beyond criticality.* While this thesis focused on conformal field theories at criticality, many physical phenomena of interest occur in theories deformed away from the critical point. Extending the non-perturbative methods developed here to non-conformal theories or massive deformations of CFTs would be a major advancement. Such an extension would open new avenues for understanding finite-temperature dynamics in realistic models and could provide deeper insights into order-disorder phase transitions [30–32] and the thermodynamic structure of quantum field theories.
- ⋆ *Line defects and non-local observables.* Finally, the techniques presented in this thesis can be adapted to study line defects wrapping the thermal circle, such as Polyakov loops, which serve as order parameters for confinement-deconfinement transitions. In particular, this would enable a rigorous study of interesting physical defects, such as magnetic line defects in $O(N)$ models [33].

[1] J. Barrat, D. N. Bozkurt, E. Marchetto, A. Miscioscia, and E. Pomoni, The analytic bootstrap at finite temperature, (2025), arXiv:2506.06422 [hep-th].

[2] J. Barrat, E. Marchetto, A. Miscioscia, and E. Pomoni, Thermal Bootstrap for the Critical O(N) Model, Phys. Rev. Lett. **134**, 211604 (2025), arXiv:2411.00978 [hep-th].

[3] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998), arXiv:hep-th/9711200.

[4] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998), arXiv:hep-th/9803131.

[5] E. Katz, S. Sachdev, E. S. Sørensen, and W. Witczak-Krempa, Conformal field theories at nonzero temperature: Operator product expansions, Monte Carlo, and holography, Phys. Rev. B **90**, 245109 (2014), arXiv:1409.3841 [cond-mat.str-el].

[6] J. Barrat, D. N. Bozkurt, E. Marchetto, A. Miscioscia, and E. Pomoni, Analytic thermal bootstrap meets holography, (2025), arXiv:2510.20894 [hep-th].

[7] I. Burić, I. Gusev, and A. Parnachev, Holographic correlators from thermal bootstrap, (2025), arXiv:2508.08373 [hep-th].

[8] S. El-Showk and K. Papadodimas, Emergent Spacetime and Holographic CFTs, JHEP **10**, 106, arXiv:1101.4163 [hep-th].

[9] L. Iliesiu, M. Koloğlu, R. Mahajan, E. Perlmutter, and D. Simmons-Duffin, The Conformal Bootstrap at Finite Temperature, JHEP **10**, 070, arXiv:1802.10266 [hep-th].

[10] L. Iliesiu, M. Koloğlu, and D. Simmons-Duffin, Bootstrapping the 3d Ising model at finite temperature, JHEP **12**, 072, arXiv:1811.05451 [hep-th].

[11] D. Carmi and S. Caron-Huot, A Conformal Dispersion Relation: Correlations from Absorption, JHEP **09**, 009, arXiv:1910.12123 [hep-th].

[12] D. Carmi, J. Moreno, and S. Sukholuski, A conformal dispersion relation for mixed correlators, (2025), arXiv:2503.08774 [hep-th].

[13] S. Caron-Huot, Analyticity in Spin in Conformal Theories, JHEP **09**, 078, arXiv:1703.00278 [hep-th].

[14] L. F. Alday, M. Kologlu, and A. Zhiboedov, Holographic correlators at finite temperature, JHEP **06**, 082, arXiv:2009.10062 [hep-th].

[15] M. Schottenloher, ed., *A mathematical introduction to conformal field theory* (Springer Berlin, Heidelberg, 2008).

[16] D. Simmons-Duffin, TASI lectures on the conformal bootstrap, (2016), arXiv:1602.07982 [hep-th].

[17] Z. Komargodski and A. Zhiboedov, Convexity and Liberation at Large Spin, JHEP **11**, 140, arXiv:1212.4103 [hep-th].

[18] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, The Analytic Bootstrap and AdS Superhorizon Locality, JHEP **12**, 004, arXiv:1212.3616 [hep-th].

[19] D. Simmons-Duffin, The lightcone bootstrap and the spectrum of the 3d ising cft, JHEP **03**, 086, arXiv:1612.08471 [hep-th].

[20] T. Matsubara, A new approach to quantum-statistical mechanics, Progress of Theoretical Physics **14**, 351 (1955).

[21] A. Das, *Finite Temperature Field Theory*, 2nd ed. (WORLD SCIENTIFIC, 2023) https://www.worldscientific.com/doi/pdf/10.1142/13308.

[22] R. Kubo, Statistical mechanical theory of irreversible processes. 1. General theory and simple applications in magnetic and conduction problems, J. Phys. Soc. Jap. **12**, 570 (1957).

[23] P. C. Martin and J. S. Schwinger, Theory of many particle systems. 1., Phys. Rev. **115**, 1342 (1959).

[24] E. Marchetto, A. Miscioscia, and E. Pomoni, Broken (super) conformal Ward identities at finite temperature, JHEP **12**, 186, arXiv:2306.12417 [hep-th].

[25] K. G. Wilson and M. E. Fisher, Critical exponents in 3.99 dimensions, Phys. Rev. Lett. **28**, 240 (1972).

[26] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4-theories* (World Scientific, 2001).

[27] E. Marchetto, A. Miscioscia, and E. Pomoni, Sum rules & Tauberian theorems at finite temperature, JHEP **09**, 044, arXiv:2312.13030 [hep-th].

[28] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Large N field theories, string theory and gravity, Physics Reports **323**, 183 (2000).

[29] M. DodelsonAnalyticity in Spin in Conformal Theories, A. Grassi, C. Iossa, D. Panea Lichtig, and A. Zhiboedov, Holographic thermal correlators from supersymmetric instantons, SciPost Phys. **14**, 116 (2023), arXiv:2206.07720 [hep-th].

[30] N. Chai, S. Chaudhuri, C. Choi, Z. Komargodski, E. Rabinovici, and M. Smolkin, Symmetry Breaking at All Temperatures, Phys. Rev. Lett. **125**, 131603 (2020).

[31] Z. Komargodski and F. K. Popov, Temperature-resistant order in $2 + 1$ dimensions, Phys. Rev. Lett. **135**, 091602 (2025).

[32] Y. Han, X. Huang, Z. Komargodski, A. Lucas, and F. K. Popov, Entropic order, Nature Communications **17**, 87 (2025).

[33] J. Barrat, B. Fiol, E. Marchetto, A. Miscioscia, and E. Pomoni, Conformal line defects at finite temperature, SciPost Phys. **18**, 018 (2025), arXiv:2407.14600 [hep-th].